This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

PLANE GEOMETRY

SUGGESTIVE METHOD

AVERY

BENJ. H. SANBORN & CO.
BOSTON NEW YORK CHICAGO

Educ T 149.03.160

Harvard College
Library

By Exchange

3 2044 097 045 934

PLANE GEOMETRY

BY THE

SUGGESTIVE METHOD

BY

JOHN A. AVERY
HEAD OF MATHEMATICS DEPARTMENT, ENGLISH HIGH SCHOOL
SOMERVILLE, MASSACHUSETTS

οὐ πόλλ' ἀλλὰ πολύ

BENJ. H. SANBORN & CO.
BOSTON, U.S.A.

Educ T 149.03.160

HARVARD COLLEGE
June 11, 1928
LIBRARY

By exchange.
Fordham University

Copyright, 1903,
By JOHN A. AVERY.

Norwood Press
J. S. Cushing & Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

THIS book is designed for the use of beginners in geometry in secondary schools, and has been used for several years in a simpler form by the teachers of mathematics in the Somerville (Mass.) English High School. No attempt has been made to depart in the least from the subject-matter given in the regular Euclidian geometries; the variation from the well-known texts has been more in regard to the presentation of the theorems and the method of development. To those teachers who feel the need of a book intermediate in treatment between a syllabus and a geometry of completely demonstrated theorems, this work is confidently submitted.

The most prominent characteristic of this book is that it is a *working* book, — a book in the use of which the pupil *is obliged* to show the result of individual effort. The distinctive features that bring about this result and that also differentiate this book from any yet published are the following: —

(1) **The development of definitions.** — To prevent the perfunctory memorizing of definitions without a thorough understanding of their meanings, the various geometrical terms and concepts are to be developed in the class as they are needed. This may be accomplished by encouraging the pupils not only to present to the class their own ideas regarding the various terms, but also to bring to the recitation definitions which they have obtained from various sources. Under the careful guidance of the teacher, satisfactory definitions may thus be decided upon and then recorded by each pupil in a note-book kept for the purpose.

(2) **Suggested demonstrations.** — Demonstrations are not given in full, but are outlined by means of hints or suggestions. Although consider-

able assistance must necessarily be given in the development of the first few theorems, it is expected that the suggestions for the remaining theorems will be, in general, sufficient guides to enable the pupil to work out his own proofs. Constant watchfulness will be necessary on the part of the teacher in order to secure logical demonstrations. The pupil must keep constantly in mind that a geometrical demonstration is a series of steps or statements leading in logical order from the hypothesis to the required conclusion, — each step being dependent on some geometrical fact already known, that is, a definition, an axiom, or some theorem previously proved. That reasons may be *readily* and *accurately* given, definitions and statements of theorems should be committed to memory.

(3) **The construction of the figures by the pupil.** — Space for the figure is provided under each theorem. The appropriate figure is to be constructed and lettered by the pupil in accordance with the directions in the hypothesis and the accompanying suggestions. Especial emphasis is laid on the correct construction of figures in order to fix firmly in the mind of the pupil the exact conditions under which the theorems are to be proved. To prevent the incorrect drawing of figures, the class and the teacher should determine exactly what each figure is to be before it is constructed in the book.

(4) **The use of the problems of construction.** — Problems in construction are given whenever needed as an assistance in the proper construction of the figures for the theorems. It follows that the method of proof for a problem cannot always be explained when the problem is given. In the review, however, the explanation of all problems should be thoroughly understood.

(5) **The choice and the arrangement of the theorems.** — In regard to the choice of theorems, effort has been made to select the theorems that will give a clear and comprehensive view of the subject. The Syllabus of Propositions prepared by the Mathematical Department of Harvard University has been used as a basis of selection, but has not been followed exactly. The arrangement of theorems, although strictly logical, has been purposely varied from the arrangements in the best known

text-books that the pupil may be as independent of outside assistance as possible.

(6) **The use of exercises.** — Over one hundred and fifty exercises are given. These furnish a minor feature only, but are of special value in the review.

This book differs radically from the geometries used in the majority of secondary schools. It is the conviction, however, of those who know its methods and have seen its practical working that the use of the book will infuse new life in the study of geometry, awaken greater interest in both teacher and pupil, secure a great gain in the amount of individual effort, increase the spirit of self-reliance, and greatly strengthen the power of deductive reasoning.

Grateful acknowledgments are due to associate teachers and others for aid and suggestions in the preparation of this book.

J. A. A.

SOMERVILLE, MASS.,
February, 1903.

CONTENTS

GEOMETRY

PAGE
TOPICS FOR DEFINITION 1
AXIOMS 3
ABBREVIATIONS 4
SYMBOLS 4

PLANE GEOMETRY

Book I. Rectilinear Figures 5
Book II. Topics for Definition 46
The Circle 47
Book III. Theory of Proportion 67
Proportional Lines and Similar Figures . . . 69
Book IV. Areas and Ratios of Polygons 84
Book V. Regular Polygons and the Circle 95
INDEX TO PROBLEMS OF CONSTRUCTION 109
SYLLABUS OF THEOREMS AND COROLLARIES 111

GEOMETRY

INTRODUCTORY TOPICS FOR DEFINITION

1. **Space.**

2. **Geometrical Body or Solid.**
   Properties, Dimensions, Parts, Kinds.

3. **Surface.**
   Dimensions.

4. **Line.**
   Dimension.

5. **Point.**

6. **Concept.**
   Space Concepts, Space Figures, Space Magnitudes.

7. **Generation.**
   Generation of Line, of Surface, of Solid.

8. **Geometry.**
   Purposes, Parts.

9. **Lines.**
   (*a*) *Kinds:*
      Straight, Curved, Broken.
   (*b*) *Positions:*
      Vertical, Horizontal, Inclined.
   (*c*) *Relative positions of two or more lines:*
      Perpendicular, Parallel, Inclined.

10. Angles.

(*a*) ***Method of Reading.***

(*b*) ***Kinds:***

(1) Right, (3) Straight, (5) Concave,
(2) Acute, (4) Obtuse, (6) Convex.

(*c*) ***Measurement.***

(*d*) ***Relation between angles:***

(1) Complementary, (4) Complementary Adjacent,
(2) Supplementary, (5) Supplementary Adjacent,
(3) Adjacent, (6) Vertical.

11. Plane Surface or Plane.

Plane Figure, Polygon, Triangle.

12. Triangle.

(*a*) ***Kinds:***

(1) Equilateral, (4) Equiangular, (6) Obtuse,
(2) Isosceles, (5) Acute, (7) Right.
(3) Scalene,

(*b*) ***Parts:***

(1) Base, (4) Perimeter,
(2) Vertex, (5) Hypotenuse (in right triangle).
(3) Altitude,

(*c*) ***Relation of triangles:***

(1) Equal, (4) Mutually Equiangular,
(2) Equivalent, (5) Mutually Equilateral.
(3) Similar,

13. Proposition.

Theorem.

Parts: Hypothesis, Conclusion.
(1) Converse, (3) Corollary,
(2) Opposite, (4) Scholium.

Problem.

14. Axioms, definition.

1. Through two given points only one straight line can be drawn.
2. A straight line is the shortest line between two points.
3. Two magnitudes equal to the same magnitude or equal magnitudes are equal.
4. If the same operation be performed on equals, the results are equal.
5. If the same operation be performed on unequals, the results are unequal in the same sense.
6. If unequals be subtracted from equals, the results are unequal in the inverse sense.
7. The whole is greater than any of its parts.
8. The whole equals the sum of its parts.
9. If, of three quantities, the first is greater than the second, and the second is greater than the third, then the first is greater than the third.
10. If the greater of two magnitudes is continuously decreasing, and the smaller continuously increasing, once, and only once, will the magnitudes be equal.
11. If two plane figures can be made to coincide by superposition, the figures are equal.

15. ABBREVIATIONS.

Adj.	adjacent.
Alt.	alternate.
Auth.	authority.
Ax.	axiom.
Comp.	complementary.
Cons.	construction.
Cor.	corollary.
Def.	definition.
Ext.	exterior.
Hom.	homologous.
Hyp.	hypothesis.
Hypo.	hypotenuse.
Iden.	identical.
Int.	interior.
Isos.	isosceles.
Opp.	opposite.
Prob.	problem.
Prop.	proposition.
Pt.	point.
Q.E.D.	quod erat demonstrandum (which was to be proved).
Q.E.F.	quod erat faciendum (which was to be done).
Quad.	quadrilateral.
Rect.	rectangle.
Rt.	right.
Scho.	scholium.
Sug.	suggestion.
Supp.	supplementary.
St.	straight.
Theo.	theorem.

16. SYMBOLS

+	plus.
−	minus.
×, ·	times.
=	equals.
≡	is equivalent to.
∼	is similar to.
>	is greater than.
<	is less than.
∴	therefore.
∠ ∠s	angle, angles.
△ △s	triangle, triangles.
⊥	perpendicular, or is perpendicular to.
⊥s	perpendiculars.
▱ ▱s	parallelogram, parallelograms.
∥	parallel, or is parallel to.
∥s	parallels.
⊙ ⓢ	circle, circles.
⌒ ⌒s	arc, arcs.

PLANE GEOMETRY

BOOK I

LINES AND RECTILINEAR FIGURES

THEOREM I

17. ***From a point in a straight line one perpendicular to the line, and only one, can be drawn.***

Hypothesis. Let C be any point in the line AB.

To prove that one perpendicular, and only one, can be drawn to AB at C.

SUGGESTION 1. Revolve a straight line, CX, about C as a centre, comparing $\angle$s BCX and XCC.

SUG. 2. Apply Axiom 10 and definition of perpendicular.

Conclusion.

18. Problem I. To draw a perpendicular to a straight line at some point in it.

(Two figures: first, perpendicular at some point within the line; second, at the extremity of the line.)

THEOREM II

19. *All right angles are equal.*

Hyp. Let $\angle CDB$ and $\angle C'D'B'$ be any two rt. $\angle$s.

To prove that $\angle CDB = \angle C'D'B'$.

SUG. 1. Place $\angle CDB$ on $\angle C'D'B'$ so that the vertex D falls on the vertex D', DB taking the direction of $D'B'$.

SUG. 2. What direction will DC take? Why?

Conclusion.

20. Problem II. To bisect a given angle.

THEOREM III

21. *If one straight line meets another, the sum of the adjacent angles thus formed equals two right angles.*

Hyp. Let DB meet CA at B.

To prove that $\angle ABD + \angle DBC = 2$ rt. $\angle$s.

SUG. 1. Draw $PB \perp AC$ at B.

SUG. 2. Compare the sum of $\angle ABP$ and $\angle PBC$ with the sum of $\angle ABD$ and $\angle DBC$.

Conclusion.

22. Corollary 1. *The sum of all the angles about a point on one side of a straight line equals two right angles.*

SUG. Group all of the angles into two angles and apply Theo. III.

23. Cor. 2. *The sum of all the angles about a point equals four right angles.*

Sug. Produce one line through the point and apply Cor. 1.

24. Cor. 3. *Complements of the same angle or of equal angles are equal.*

25. Cor. 4. *Supplements of the same angle or of equal angles are equal.*

THEOREM IV

26. ***If two straight lines intersect, they form vertical angles which are equal.***

Hyp. Let AB intersect CD at X. (See figure on next page.)

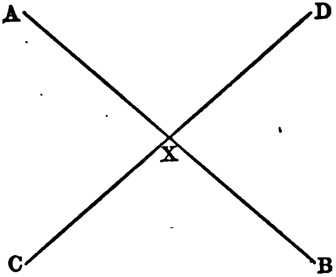
To prove that $\angle AXC = \angle BXD$, also that $\angle DXA = \angle CXB$.

Sug. 1. What relation is there between $\angle AXC$ and $\angle DXA$?
Sug. 2. What relation is there between $\angle BXD$ and $\angle DXA$?
Sug. 3. Then what follows regarding $\angle AXC$ and $\angle BXD$?
Sug. 4. Follow same method with $\angle AXD$ and $\angle CXB$.
Conclusion.

Note. On the next page will be found a **model proof** of Theo. IV showing an acceptable method for arranging and expressing a geometrical demonstration.

MODEL PROOF THEOREM IV

If two straight lines intersect, they form vertical angles which are equal.



Hyp. Let AB intersect CD at X.

To prove that $\angle AXC = \angle BXD$, also that $\angle DXA = \angle CXB$.

Proof. $\angle AXC + \angle DXA = 2$ rt. $\measuredangle$s. Theo. III.

$\angle BXD + \angle DXA = 2$ rt. $\measuredangle$s. Theo. III.

$\therefore \angle AXC = \angle BXD$. Theo. III., Cor. 4.

In like manner $\angle AXD = \angle CXB$. Q.E.D.

EXERCISES

1. What is the complement of an angle of 37° 57′? the supplement?

2. In figure under Theo. IV, draw a line, MN, through X bisecting $\angle AXC$. Prove that the line produced bisects $\angle DXB$ also.

3. State Exercise 2 in the form of a theorem.

4. If $\angle AXC = 73° 47' 3''$, find the value of $\angle AXD$, of $\angle DXB$, and of $\angle CXB$.

5. Prove that a line through X perpendicular to MN bisects the $\angle AXD$.

Theorem V

27. ***If two adjacent angles are together equal to two right angles, their exterior sides are in the same straight line.***

Hyp. Let $\angle ABC + \angle CBD = 2$ rt. $\angle$s.

To prove that AB and BD form one st. line.

Sug. 1. Imagine AB produced to X.
Sug. 2. $\angle ABC + \angle CBX =$ what? Auth.
Sug. 3. Compare this equation with the hypothesis.
Sug. 4. What relation must follow between $\angle CBD$ and $\angle CBX$? between BD and BX? Auth.
Conclusion.

28. Problem III. To construct an angle equal to a given angle.

THEOREM VI

29. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal.*

Hyp. In the $\triangle\!\triangle$ ABC and DEF, let $AC = DF$, $AB = DE$, $\angle A = \angle D$.

To prove that $\triangle ABC = \triangle DEF$.

SUG. 1. Apply $\triangle ABC$ to $\triangle DEF$ so that AC coincides with DF. Why is this possible?

SUG. 2. What direction will AB take? Why? Where will B fall? Why?

SUG. 3. What will follow regarding BC and EF?

Conclusion.

30. Scholium I. When two figures have been made to coincide *by superposition*, their parts thus shown to be equal are called homologous parts. It follows therefore that *homologous parts of equal figures are equal.*

31. Scho. II. In equal triangles, equal sides lie opposite equal angles and equal angles lie opposite equal sides.

NOTE. The method of proof employed in Theos. II and VI is called the *method of superposition.*

THEOREM VII

32. ***If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal.***

Hyp. In the $\triangle$ ABC and DEF, let $AC = DF$, $\angle A = \angle D$, $\angle C = \angle F$.

To prove that $\triangle ABC = \triangle DEF$.

SUG. 1. See Sug. 1 in preceding theorem.
SUG. 2. What direction will AB take? Why?
SUG. 3. B, the extremity of AB, will fall where?
SUG. 4. What direction will BC take? Why?
SUG. 5. B, the extremity of BC, will fall where?
SUG. 6. Why, then, must B fall exactly on E?
Conclusion.

33. Problem IV. To bisect a given straight line.

THEOREM VIII

34. *The angles opposite the equal sides of an isosceles triangle are equal.*

Hyp. Let AB and AC be the equal sides of the isos. $\triangle ABC$.

To prove that $\angle B = \angle C$.

SUG. 1. Draw a line bisecting $\angle A$ and meeting BC at M.
SUG. 2. Compare $\triangle$s ABM and ACM. Auth.
SUG. 3. Compare $\angle$s B and C, using scholiums under Theo. VI.
Conclusion.

35. Cor. *An equilateral triangle is also equiangular.*

EXERCISES

6. Prove Theo. VIII by connecting A with middle point of BC and comparing the triangles thus formed.

7. If the equal sides of an isosceles triangle are produced beyond the base, the angles formed are equal.

8. How many degrees in an angle whose complement and supplement together equal 210°?

9. In the isosceles triangle ABC, let $AB = AC$. From H, middle point of AB, draw HC; from K, middle point of AC, draw KB. Prove $HC = KB$.

SUG. Compare $\triangle$s AHC and AKB.

THEOREM IX

36. *If two triangles have three sides of one equal respectively to three sides of the other, the triangles are equal.*

Hyp. In $\triangle$s ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, $BC = B'C'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

SUG. 1. Place $\triangle ABC$ in the plane of $\triangle A'B'C'$ with AB on its equal $A'B'$, and with C on the opposite side of AB from C'. Draw CC'.

SUG. 2. What kind of $\triangle$ is $A'CC'$? $B'CC'$?

SUG. 3. Then compare $\angle$s $A'C'C$ and $C'CA'$; $\angle$s $CC'B'$ and $B'CC'$.

SUG. 4. Then compare $\angle$s C and C'. Auth.

SUG. 5. Compare $\triangle$s ABC and $A'B'C'$. Auth.

Conclusion.

EXERCISES

10. Are two triangles equal if the three angles of one are equal respectively to the three angles of the other?

11. Can two triangles, having two sides and an angle of one equal respectively to two sides and an angle of the other, be unequal?

12. Four of the five consecutive angles about a point contain 17°, 36°, 89°, nd 110° respectively. How many degrees are there in the fifth angle?

THEOREM X

37. *If a perpendicular be erected at the middle point of a straight line, the distances from any point in the perpendicular to the extremities of the line are equal.*

Hyp. Let PB be $\perp$ MR at middle pt. B. Let C be any pt. in PB.

To prove that $CM = CR$.

SUG. 1. Compare $\triangle$s CMB and CRB. Auth.

SUG. 2. Compare CM and CR, using Scholiums under Theo. VI.

Conclusion.

38. Scho. Any side of a triangle is less than the sum of the other two sides.

EXERCISES

13. The bisector of the vertical angle of an isosceles triangle is perpendicular to the base.

14. If the angular magnitude about a point is divided into three angles, the second of which is twice the first, and the third is three times the first, how many degrees are there in each of the three angles?

15. Any side of a triangle is less than half the sum of its three sides.

THEOREM XI

39. *If a perpendicular be erected at the middle point of a straight line, the distances from any point not in the perpendicular to the extremities of the line are unequal.*

Hyp. Let KD be $\perp MB$ at the middle pt. D. From F, any pt. outside KD, draw FM and FB. Let FM intersect KD at C.

To prove that $FM > FB$.

SUG. 1. Draw CB. Compare MC and CB. Auth.
SUG. 2. Compare FB with $FC + CB$. Auth.
SUG. 3. Compare FB with $FC + CM$.
Conclusion.

40. Problem V. Given two sides and the angle to be included by them, to construct the triangle.

THEOREM XII

41. *From a point without a straight line, one perpendicular to the line, and only one, can be drawn.*

PART 1. **Hyp.** Let AB be the line, C any pt. not in AB.

To prove that one $\perp$ can be drawn from C to AB.

SUG. 1. To a st. line MN erect a $\perp$ KL at any point in the line, as L. Auth.

SUG. 2. Apply this figure so that MN falls on AB, and move it along AB till KL includes C. What conclusion follows?

PART 2. **Hyp.** Let CL be $\perp AB$.

To prove that no other $\perp$ can be drawn to AB from C.

SUG. 1. Draw a line CD assuming it to be also $\perp AB$. Extend CL to C', making $LC' = CL$. Draw $C'D$.

SUG. 2. Compare $\triangle$s CLD and $C'LD$. Auth. Compare $\angle$s CDL and LDC'.

SUG. 3. If $CD \perp AB$, $\angle CDL$ is a rt. $\angle$. What follows regarding $\angle LDC'$?

SUG. 4. From the conclusions in Sug. 3, what kind of line has CDC' been proved to be? Auth.

SUG. 5. If this last conclusion is incorrect (why?) what previous statement, upon which this conclusion was based, must have been wrong?

Conclusion.

THEOREM XIII

42. *If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, the triangles are equal.*

Hyp. In the rt. $\triangle$s ABC and $A'B'C'$, let $AC = A'C'$, $\angle A = \angle A'$, $\angle B$ and $\angle B'$ being rt. $\angle$s.

To prove that $\triangle ABC = \triangle A'B'C'$.

SUG. 1. Place $\triangle ABC$ on $\triangle A'B'C'$ so that AC coincides with $A'C'$. What direction will AB take? Why?

SUG. 2. Why must CB and $C'B'$ coincide in direction?

Conclusion.

43. Problem VI. Given two angles and the side to be included by them, to construct the triangle.

THEOREM XIV

44. *Two lines perpendicular to the same line are parallel.*

Hyp. Let RD and FG be $\perp$ KL at D and G respectively.

To prove that $RD \parallel FG$.

SUG. 1. If RD and FG are not $\parallel$, they must meet at some point.
SUG. 2. Compare this conclusion with Theo. XII.
Conclusion.

45. Ax. 12. Through a given point only one straight line can be drawn parallel to a given line.

46. Problem VII. Given three sides of a triangle, to construct the triangle.

47. Problem VIII. To draw a line parallel to a given line through a given point.

THEOREM XV

48. *A line perpendicular to one of two parallel lines is perpendicular to the other also.*

Hyp. Let *RC* be ∥ *FD*, *RC* ⊥ *KL*.

To prove that *FD* ⊥ *KL*.

Sug. 1. From *Q*, some pt. in *FD*, imagine *QM* drawn ⊥ *KL*.
Sug. 2. What relation has *QM* to *RC*? Auth.
Sug. 3. What must follow regarding *QM* and *FD*?
Conclusion.

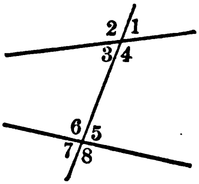
49. Cor. *Two lines parallel to a third line are parallel to each other.*

Sug. Draw a line perpendicular to a third line.
Apply Theos. XV and XIV.

50. Problem IX. To draw a line perpendicular to a given line from an outside point.

51. Definition. If two straight lines are cut by a third line, called the **transversal**, eight angles are formed.

These angles are called, as follows: —



(1) Interior angles, 3, 4, 5, 6.
(2) Exterior angles, 1, 2; 7, 8.
(3) Alternate-interior angles, 3, 5; 4, 6.
(4) Alternate-exterior angles, 1, 7; 2, 8.
(5) Corresponding angles, 2, 6; 1, 5;
3, 7; 4, 8.

THEOREM XVI

52. *If two parallel lines are cut by a transversal, the alternate-interior angles are equal.*

Hyp. Let the $\parallel$ lines MN and RP be cut by the transversal XY at F and G respectively.

To prove that $\angle MFG = \angle PGF$, $\angle GFN = \angle FGR$.

SUG. 1. From S, middle pt. of FG, draw $SC \perp MN$ and produced to meet RP at D. What relation has CD to RP? Auth.

SUG. 2. Compare $\triangle$s CFS and SGD. Auth.

SUG. 3. What follows regarding $\angle$s CFS and DGS?

SUG. 4. Compare $\angle$s GFN and FGR, taking into account their relation to $\angle$s MFG and PGF respectively.

Conclusion.

Theorem XVII

53. ***If two parallel lines are cut by a transversal, the corresponding angles are equal.***

Hyp. See hypothesis under Theo. XVI.

To prove that $\angle NFX = \angle PGF$, etc.

Sug. Consult Theos. IV and XVI.

Conclusion.

54. Cor. *If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.*

Theorem XVIII

55. ***If two parallel lines are cut by a transversal, the sum of the interior angles on the same side of the transversal equals two right angles.***

Note. See figure for Theo. XVII.

Hyp. See hyp. under Theo. XVI.

To prove that $\angle GFN + \angle PGF = 2$ rt. $\angle$s.

Sug. Consult Theos. III and XVI.

Conclusion.

56. Cor. *If two parallel lines are cut by a transversal, the sum of the exterior angles on the same side of the transversal equals two right angles.*

THEOREM XIX

57. (Converse of Theo. XVI.) *If two straight lines are cut by a transversal, so that the alternate-interior angles are equal, the lines are parallel.*

Hyp. Let AB and LM be cut by XY at G and H respectively, so that $\angle HGB = \angle GHL$.

To prove that $AB \parallel LM$.

SUG. 1. If AB is not $\parallel LM$, draw CD through G, assuming it to be $\parallel LM$.

SUG. 2. Compare $\angle$s HGD and GHL, and then compare the conclusion with the equation in the Hyp.

SUG. 3. What follows regarding $\angle$s HGB and HGD? Auth.

SUG. 4. What follows regarding lines AB and CD? Auth.

SUG. 5. What follows regarding lines AB and LM?

Conclusion.

EXERCISES

16. Given two parallel lines cut by a transversal, name eight pairs of angles whose sum equals two right angles.

17. Given two parallel lines cut by a transversal, prove that the bisectors of a pair of alternate-interior angles are parallel.

THEOREM XX

58. (Converse of Theo. XVII.) *If two straight lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel.*

Hyp. Let XY and ZW be cut by BC at D and K respectively, so that $\angle YDB = \angle WKD$.

To prove that $XY \parallel ZW$.

SUG. 1. If XY is not $\parallel$ ZW, draw RS through D, assuming it to be $\parallel$ ZW.

SUG. 2. Use method suggested in Theo. XIX.

Conclusion.

59. Cor. *If two straight lines are cut by a transversal so that the alternate-exterior angles are equal, the lines are parallel.*

EXERCISES

18. Prove corollary under Theo. XV by drawing a transversal intersecting the three lines, then using a method suggested by Theos. XVI to XX.

19. A straight line parallel to the base of an isosceles triangle makes equal angles with the other two sides.

20. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

THEOREM XXI

60. (Converse of Theo. XVIII.) ***If two straight lines are cut by a transversal so that the sum of the interior angles on the same side of the transversal equals two right angles, the lines are parallel.***

Hyp. Let XY and ZW be cut by BC at D and K respectively, so that $\angle KDY + \angle WKD = 2$ rt. $\angle$s.

To prove $XY \parallel ZW$.

Sug. 1. If XY is not $\parallel ZW$, draw RS through D, assuming it to be $\parallel ZW$.

Sug. 2. Use method suggested in Theo. XIX.

Conclusion.

61. Cor. *If two straight lines are cut by a transversal so that the sum of the exterior angles on the same side of the transversal equals two right angles, the lines are parallel.*

EXERCISES

21. Two straight lines AB and CD bisect each other at O. Show that the straight lines joining AC and BD are parallel.

22. The bisectors of two supplementary adjacent angles are perpendicular to each other.

23. State and prove the converse of Ex. 22.

THEOREM XXII

62. *Angles whose sides are parallel, each to each, are either equal or supplementary.*

Hyp. Let BA be $\parallel NR$, $BC \parallel MT$, and let NR and MT intersect at S.

NOTE. At first, for convenience, let $\angle CBA$ be drawn above and to the left of $\angle TSR$ and let $\angle CBA$ be an acute angle.

To prove that $\angle CBA = \angle TSR = \angle MSN$ and also that $\angle CBA$ is the supplement of $\angle RSM$ and of $\angle NST$.

SUG. 1. Let BC and NR intersect at X.

SUG. 2. Compare $\angle$s CBA and CXR. Auth.

SUG. 3. Compare $\angle$s CXR and TSR. Auth.

First conclusion.

SUG. 4. $\angle TSR$, and therefore its equal $\angle CBA$, has what relation to $\angle RSM$?

Second conclusion.

63. Scho. Angles whose sides are parallel are *equal* when both pairs of parallel sides extend in the same direction, or in opposite directions from the vertices; *supplementary*, when one pair of parallel sides extends in the same direction and the other pair in opposite directions from the vertices.

64. Cor. *Angles whose sides are perpendicular, each to each, are either equal or supplementary.*

SUG. Through the vertex of one angle draw lines parallel to the sides of the other and extending in the same direction.

65. Topic for Definition. Exterior angle of a triangle.

THEOREM XXIII

66. *An exterior angle of a triangle equals the sum of the opposite interior angles.*

NOTE. **Put figures for Theo. XXIII and Theo. XXIV in this space.**

Hyp. Let $\angle ACD$ be an ext. $\angle$ of the $\triangle ABC$.

To prove that $\angle ACD = \angle A + \angle B$.

SUG. 1. Through C draw $CE \parallel AB$.

SUG. 2. Compare $\measuredangle s$ ACE and ECD with $\measuredangle s$ A and B respectively.

SUG. 3. Compare sum of $\measuredangle s$ ACE and ECD with the sum of $\measuredangle s$ A and B.

Conclusion.

67. Cor. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

THEOREM XXIV

68. *The sum of the interior angles of a triangle equals two right angles.*

Hyp. Let $\triangle ABC$ be the given triangle.

To prove that $\angle A + \angle B + \angle C = 2$ rt. $\measuredangle s$.

SUG. Produce one side of $\triangle$ and apply Theos. III and XXIII.

Conclusion.

69. Cor. 1. *In a triangle there can be only one obtuse or one right angle.*

70. Cor. 2. *Every right triangle has two acute angles, each of which is the complement of the other.*

71. Cor. 3. *If two triangles have two angles of one equal to two angles of the other, the third angles are equal.*

72. Cor. 4. *If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.*

EXERCISES

24. Prove Theo. XXIV by drawing a line through the vertex parallel to the base.

25. If an exterior angle at the base of an isosceles triangle equals 108°, how many degrees are there in the angle at the vertex?

26. What is the value of each angle of an equiangular triangle? of an equilateral triangle?

27. The angle B of a triangle ABC is three times the angle A, and the angle C is five times the angle A; how many degrees are there in each angle?

28. If two angles of a triangle are 50° and 70°, state values for all interior and exterior angles of the triangle.

29. If two angles of a triangle are 50° and 70°, how many degrees in the angle formed by the bisectors of these angles?

30. The sum of the three exterior angles of a triangle formed by producing the sides in succession is equal to four right angles.

31. If two parallel lines are cut by a transversal, the bisectors of the interior angles on the same side of the transversal are perpendicular to each other.

32. The bisector of any exterior angle of an equiangular triangle is parallel to the opposite side of the triangle.

33. If from any point in the bisector of an angle a parallel to one of the sides be drawn, an isosceles triangle is formed by the bisector, the parallel, and the remaining side of the angle.

THEOREM XXV

73. *If two right triangles have a side and an acute angle of one equal respectively to a side and an acute angle of the other, the triangles are equal.*

Hyp. In the rt. $\triangle s$ ABC and RST, let rt. $\angle B =$ rt. $\angle S$, $\angle C = \angle T$, $AB = RS$.

To prove $\triangle ABC = \triangle RST$.

SUG. Apply Theo. XXIV, Cor. 4, and Theo. VII.

Conclusion.

THEOREM XXVI

74. *If two triangles have two angles of one equal respectively to two angles of the other and the sides opposite a pair of equal angles equal, the triangles are equal.*

Hyp. In the $\triangle s$ XYZ and RST, let $\angle X = \angle R$, $\angle Z = \angle T$, and side $XY =$ side RS.

To prove $\triangle XYZ = \triangle RST$.

SUG. See Theo. XXIV, Cor. 3.

Conclusion.

THEOREM XXVII

75. (Converse of Theo. VIII.) *If two angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.*

Hyp. In $\triangle ABC$, let $\angle A = \angle B$.

To prove that $AC = BC$, or that $\triangle ABC$ is isos.

SUG. 1. Draw $CD \perp AB$.
SUG. 2. Compare $\triangle$s ACD and BCD.
Conclusion.

76. Cor. *An equiangular triangle is also equilateral.*

EXERCISES

34. Prove Theo. XXV if $\angle A = \angle R$ and side $BC =$ side ST.

35. If the angle at the vertex of an isosceles triangle is 35° 40′, how many degrees are there in either of the angles at the base?

36. Prove Theo. XXVII by drawing CD bisecting $\angle C$.

37. The bisector of the vertical angle of an isosceles triangle is perpendicular to the base. (Prove from Ex. 36.)

38. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.

39. AM is the bisector of a base angle of the isosceles triangle ABC, the bisector meeting the side BC in M; the vertical angle is 28°. How many degrees are there in angle AMC?

THEOREM XXVIII

77. *If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.*

Hyp. In $\triangle ABC$, let $\angle C$ be $> \angle A$.

To prove that $AB > BC$.

Sug. 1. Through C draw CM, making $\angle ACM = \angle MAC$.
Sug. 2. Compare AM and CM. Auth.
Sug. 3. Compare BC and $BM + CM$. Auth.
Sug. 4. Substitute result of Sug. 2 in result of Sug. 3.
Conclusion.

78. Cor. *The hypotenuse is the greatest side of a right triangle.*

79. Problem X. Given two angles of a triangle, to find the third angle.

THEOREM XXIX

80. (Converse of Theo. XXVIII.) *If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle being opposite the greater side.*

Hyp. In the $\triangle ABC$, let AB be $> BC$.

To prove that $\angle C > \angle A$.

SUG. 1. Either $\angle C = \angle A$, $\angle C < \angle A$, or $\angle C > \angle A$.

SUG. 2. Prove that the first two statements lead to results inconsistent with the hypothesis, stating authorities.

SUG. 3. If, out of three possible statements, two have been proved incorrect, what follows regarding the third?

Conclusion.

NOTE. This method of proof is commonly known as *reductio ad absurdum*, or the *indirect method*. By the direct method the hypothesis is taken as the basis and from it the conclusion is obtained by the direct application of axioms and theorems already known; by the indirect method it is assumed that the conclusion to be proved is not true, and it is then shown that this assumption leads to an absurdity or to a result inconsistent with the hypothesis. What theorems have already been proved by the indirect method?

Theorem XXX

81. *A perpendicular is the shortest line that can be drawn from a point to a line.*

Hyp. From C, any pt. outside AB, let CD be $\perp$ AB at D.

To prove that CD is the shortest line that can be drawn from C to AB.

Sug. 1. Draw CE any other line from C to AB.

Sug. 2. In $\triangle CDE$ compare $\angle D$ and $\angle E$.

Sug. 3. Then compare CE and CD. Auth.

Conclusion.

82. Cor. *The shortest distance from a point to a line is the perpendicular from the point to the line.*

EXERCISES

40. How many degrees are there in each angle of an equilateral triangle?

41. In the isosceles triangle ABC, let perpendiculars AM and BN be drawn from the vertices of the equal angles A and B to the opposite sides. Prove $AM = BN$.

THEOREM XXXI

83. *If a perpendicular and oblique lines be drawn from a point to a line:*

I. *Two oblique lines which meet the given line at equal distances from the foot of the perpendicular are equal.*

II. *Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the more remote is the greater.*

Hyp. Let SR be $\perp$ XY at S, and let RK, RN, and RL be oblique lines so drawn that $SK = SL$ and $SN > SL$.

NOTE. For convenience in following suggestions, lay off SN and SK to the left of $\perp$, SL to the right.

To prove that (1) $RK = RL$, (2) $RN > RL$.

SUG. 1. Compare $\triangle$s KRS and LRS.

First conclusion.

SUG. 2. Compare $\angle$s SKR and RLS. Auth.

SUG. 3. Compare $\angle$s SNR and SKR. Auth.

SUG. 4. Compare $\angle$s SNR and RLS. Auth.

SUG. 5. From conclusion in Sug. 4, compare RN and RL, considering them as sides of $\triangle$ RNL. Auth.

Second conclusion.

THEOREM XXXII

84. *Two equal oblique lines drawn from a point in a perpendicular to a given line, cut off equal distances from the foot of the perpendicular.*

Hyp. Let BS be $\perp$ XY at S. From N, any pt. in BS, draw NR and NT, equal oblique lines, to XY.

To prove that $RS = TS$.

SUG. Compare $\triangle$ NRS and NST.

Conclusion.

THEOREM XXXIII

85. *Two unequal oblique lines drawn from a point in a perpendicular to a given line, cut off unequal distances from the foot of the perpendicular, the longer line cutting off the greater distance.*

Hyp. Let BS be $\perp$ XY at S. Of the oblique lines NR and NT, let NR be $< NT$.

To prove that $SR > ST$.

SUG. Use method suggested in Theo. XXIX.

Conclusion.

THEOREM XXXIV

86. *If two right triangles have the hypotenuse and a side of one equal respectively to the hypotenuse and a side of the other, the triangles are equal.*

Hyp. In rt. $\triangle$ ABC and RST, let hypo. $AC =$ hypo. RT, side $AB =$ side RS.

To prove that $\triangle ABC = \triangle RST$.

Sug. 1. Place $\triangle ABC$ in plane of $\triangle RST$ so that AB coincides with RS, pt. C falling on opp. side of RS from T.

Sug. 2. What kind of line is CST? Auth.

Sug. 3. Since $CR = RT$, what follows regarding CS and ST? Auth.

Conclusion.

EXERCISES

42. In a triangle ABC, $\angle A = 2\angle B$, $\angle B = 3\angle C$. Find the value of each angle.

43. A line drawn from the vertex of the right angle of a right triangle to the middle point of the hypotenuse divides the triangle into two isosceles triangles.

44. Collect all theorems by means of which triangles in general are proved equal; by means of which right triangles are proved equal.

THEOREM XXXV

87. *If two lines be drawn from a point in a triangle to the extremities of a side, the sum of these lines is less than the sum of the other two sides of the triangle.*

Hyp. Let D be any pt. in $\triangle ABC$, DB and DC lines drawn from D to B and C respectively.

To prove that $BA + AC > BD + DC$.

SUG. 1. Produce BD to meet AC at E.

SUG. 2. Compare $BA + AE$ with BE, and then by adding EC to both sides compare $BA + AE + EC$ with $BE + EC$. Auth.

SUG. 3. Compare $DE + EC$ with DC, and then by adding BD to both sides compare $BD + DE + EC$ with $BD + DC$. Auth.

SUG. 4. Then compare $BA + AC$ with $BD + DC$.

Conclusion.

EXERCISES

45. If any point within a triangle be connected with the vertices, the sum of these lines is greater than half the perimeter of the triangle.

46. If one of the acute angles of a right triangle is $58^\circ\ 13'\ 32''$, what is the value of the other acute angle?

47. If one of the acute angles of a right triangle is twice the other, what is the value of each angle?

48. Collect all the theorems by means of which two angles may be proved either directly or indirectly equal.

Theorem XXXVI

88. *If two triangles have two sides of one equal to two sides of the other and the included angles unequal, the remaining sides are unequal, that being greater which belongs to the triangle having the greater included angle.*

Hyp. In $\triangle$s ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, $\angle A > \angle A'$.

To prove that $BC > B'C'$.

Sug. 1. Place $\triangle ABC$ in plane of $\triangle A'B'C'$ so that AB coincides with $A'B'$. What position will AC take with respect to $A'C'$? Why?

Sug. 2. Draw CC'. What kind of triangle is $\triangle C'A'C$? Compare $\angle$s $C'CA'$ and $A'C'C$.

Sug. 3. Compare $\angle$s $B'C'C$ and $C'CB'$.

Sug. 4. Compare lines $B'C$ and $B'C'$, or their equivalents BC and $B'C'$.

Conclusion.

EXERCISES

49. In Theo. XXXVI, after applying $\triangle ABC$ to $\triangle A'B'C'$ in the way mentioned, draw a line from A' bisecting $\angle C'A'C$ and meeting $B'C$ at D and draw DC'. Show that $DC = DC'$ and that $B'D + DC'$, or the equivalent $B'D + DC$, $< B'C'$.

THEOREM XXXVII

89. *If two triangles have two sides of one equal to two sides of the other and the third sides unequal, the angles opposite the third sides are unequal, that being greater which is opposite the greater third side.*

Hyp. In $\triangle$ ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, $BC > B'C'$.

To prove that $\angle A > \angle A'$.

SUG. Use method employed in Theos. XXIX and XXXIII.

Conclusion.

Topics for Definition.

90. Quadrilateral.

- Trapezium.
- Trapezoid.
  - Bases, legs, altitude.

91. Parallelogram.

  - Bases, altitude, diagonals.
- Rhomboid.
- Rectangle.
- Rhombus.
- Square.

Theorem XXXVIII

92. *The opposite sides of a parallelogram are equal.*

Hyp. Let $ABCD$ be a ▱.

Note. These letters are given in rotation *around* the figure and are read counter-clockwise.

To prove that $AB = CD$ or $AD = BC$.

Sug. 1. Draw diagonal AC.

Sug. 2. Compare △ ABC and ACD.

Conclusion.

93. Cor. *The diagonal of a parallelogram divides it into two equal triangles.*

Theorem XXXIX

94. *The opposite angles of a parallelogram are equal.*

Note. Use figure for Theo. XXXVIII.

Sug. Draw diagonal, comparing the △ formed.

Conclusion.

EXERCISES

50. Construct a parallelogram $ABCD$, given $AB = 1$ inch, $AD = 2$ inches, and $\angle A = 45°$.

51. The sum of the interior angles of a parallelogram equals four right angles.

Sug. Consult Theo. XVIII.

THEOREM XL

95. *If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.*

Hyp. In quad. $ABCD$ let $BC = AD$, let BC be ∥ AD.

To prove that $ABCD$ is a ▱.

Sug. 1. Draw diagonal AC.
Sug. 2. Compare △s ABC and ACD.
Sug. 3. Compare ∠s BAC and ACD.
Sug. 4. What follows regarding AB and CD?
Conclusion.

THEOREM XLI

96. *A quadrilateral whose opposite sides are equal is a parallelogram.*

Hyp. In quad. $ABCD$, let $BC = AD$, $AB = CD$.

To prove that $ABCD$ is a ▱.

Sug. Follow method used in Theo. XL.
Conclusion.

THEOREM XLII

97. *Two parallelograms having two sides and the included angle of one equal to two sides and the included angle of the other are equal.*

Hyp. In ▱s $ABCD$ and $A'B'C'D'$, let $AB = A'B'$, $AD = A'D'$, $\angle A = \angle A'$.

To prove that ▱ $ABCD =$ ▱ $A'B'C'D'$.

SUG. 1. Place ▱ AC on ▱ $A'C'$ so that AB coincides with $A'B'$, then AD will coincide with $A'D'$. Why?

SUG. 2. What direction will BC take? Why?

SUG. 3. What direction will DC take? Why?

Conclusion.

THEOREM XLIII

98. *The diagonals of a parallelogram bisect each other.*

Hyp. In ▱ $ABCD$, let AC and BD intersect at X.

To prove that $AX = CX$, $BX = DX$.

SUG. Compare △s AXD and BXC.

Conclusion.

EXERCISES

52. If one angle of a parallelogram is a right angle, the figure is a rectangle.

53. The line joining the middle points of the sides of a square taken in order inclose a square.

Sug. In the square *ABCD* let *E* be middle point of *AB*; *F*, of *BC*; *G*, of *CD*; and *H*, of *AD*.

Prove $\angle$s *AEH* and *BEF* each equal to 45°. $\therefore \angle HEF = 90°$. Auth. Finish proof.

Theorem XLIV

99. *The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides minus four right angles.*

Hyp. Let *A–E* be a polygon of *n* sides.

To prove that sum of int. $\angle$s of polygon $A\text{–}E = 2n$ rt. $\angle$s $- 4$ rt. $\angle$s, or $(2n-4)$ rt. $\angle$s.

Sug. 1. Connect all vertices with some pt. within, as *O*.

Sug. 2. From sum of int. $\angle$s of $\triangle$s thus formed deduct the sum of $\angle$s about *O*.

Conclusion.

100. Cor. *In an equiangular polygon of n sides, the value of each angle equals* $\dfrac{(2n-4)\text{ rt. }\angle\text{s}}{n}$.

THEOREM XLV

101. *If the sides of a polygon be produced in succession, the sum of the exterior angles thus formed equals four right angles.*

SUG. 1. What is the sum of ext. and int. $\angle$s at each vertex? at all the vertices? Auth.

SUG. 2. What is sum of int. $\angle$s alone? Auth.

SUG. 3. What then is the sum of ext. $\angle$s alone?

Conclusion.

EXERCISES

54. How many degrees are there in the interior angles of a pentagon, a hexagon, an octagon, a decagon?

55. How many degrees are there in each angle of an equiangular pentagon, hexagon, octagon, or decagon?

56. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

57. If the angles at the base of a trapezoid are equal, the non-parallel sides are equal.

58. State and prove the converse of Ex. 54.

59. The diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

60. In an isosceles trapezoid, each base makes equal angles with the legs.

61. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

62. $ABCD$ is a trapezoid whose parallel sides are AD and BC. If E and F are the middle points of AB and CD respectively, prove $EF \parallel AD$.

102. Def. The **locus** of a point is the line, or lines, which contain all the points that satisfy a given condition. In proving a theorem under **loci**, two facts must be established:

(1) that every point in the supposed locus fulfils the given condition;

(2) that no point outside the supposed locus fulfils the given condition.

Example. The locus of all points equidistant from two given points is a straight line bisecting at right angles the line joining the given points.

Hyp. Let A and B be two given pts.

To prove that XY, $\perp$ to AB at its middle pt. C, is the required locus.

Proof. (1) Every pt. in XY is equidistant from A and B.
Theo. X.

(2) Every pt. without XY is unequally distant from A and B.
Theo. XI.

$\therefore$ XY is the locus of all pts. equidistant from A and B. Q.E.D.

EXERCISES

63. What is the locus of all points at a given distance from a given line?

64. What is the locus of all points equidistant from two parallel lines?

65. The locus of all points equidistant from the sides of an angle is the bisector of the angle.

Sug. 1. Let BD be the bisector of $\angle ABC$, and from X, any pt. in BD, draw $XY \perp BC$ and $XZ \perp BA$. Prove $XY = XZ$.

Sug. 2. From K, any pt. without BD, draw $KM \perp BC$, and $KN \perp BA$ and cutting BD at P. Draw $PR \perp BC$ and RK.

$$MK < RK;\ RK < KP + RP \text{ or } RK < KP + PN.$$

What follows regarding MK and KN?

BOOK II

THE CIRCLE

ARCS, CHORDS, TANGENTS, AND MEASUREMENT OF ANGLES

Topics for Definition.

103. **Revolution.**

104. **Circle.**

- Centre.
- Radius.
- Circumference.
- Arc.
- Chord.
- Diameter.
- Secant.
- Tangent.
  - Point of tangency.
- Sector.
- Segment.
- Central angle.
- Inscribed angle.

105. Axiom 13. Radii of the same circle or of equal circles are equal.

THEOREM I

106. *The diameter of a circle is greater than any other chord; and it bisects the circle and the circumference.*

Hyp. In the $\odot ADBE$, let C be the centre, AB a diameter, and AD any other chord.

To prove first, that $AB > AD$; second, that AB bisects $\odot C$ and circumference $ADBE$.

SUG. 1. Draw radius CD. Compare AD with $AC + CD$. Auth.
SUG. 2. Compare AD with $AC + CB$.
First conclusion.
SUG. 3. Revolve segment ADB about AB into plane of segment AEB. Why will ⓢ ADB and AEB coincide?
Second conclusion.

Topics for Definition.

107. Semicircle.
Semicircumference.

EXERCISES

66. What is the locus of all the points equidistant from a given point?

67. What is the locus of all the points equidistant from the circumference of a given circle?

THEOREM II

108. *In the same circle, or in equal circles, equal central angles intercept equal arcs.*

NOTE. For convenience of construction, confine $\measuredangle$ ACB and $L'C'B'$ to upper half of Ⓢ C and C'.

Hyp. In the equal Ⓢ C and C', let $\angle ACB = \angle A'C'B'$.

To prove that $\frown AB = \frown A'B'$.

SUG. 1. Apply $\odot C$ to $\odot C'$ so that AC coincides with $A'C'$. Why is this possible? Then CB will take the direction of $C'B'$ and B will fall on B'. Why?

SUG. 2. Where will $\frown AB$ fall? Why?

Conclusion.

THEOREM III

109. (Converse of Theo. II.) *In the same circle, or in equal circles, equal arcs subtend equal central angles.*

NOTE. In constructing a figure for this theorem, use the lower half of circles constructed under Theo. II.

Hyp. In the equal Ⓢ C and C', let $\frown XY = \frown X'Y'$.

To prove that $\angle XCY = \angle X'C'Y'$.

SUG. Prove by method of superposition.

Conclusion.

110. Scho. In the same circle, or in equal circles, the greater of two central angles intercepts the *greater* arc; conversely, the greater of two arcs subtends the *greater* central angle.

THEOREM IV

111. *If, in the same circle, or in equal circles, two arcs are equal, the chords subtending them are equal.*

NOTE. Use arrangement for figures employed in Theos. II and III.

Hyp. In the equal ⓢ S and S', let $\frown XY = \frown X'Y'$.

To prove that chord $XY =$ chord $X'Y'$.

SUG. 1. Draw radii XS, YS, $X'S'$, and $Y'S'$.
SUG. 2. Compare ∠s S and S'. Auth.
SUG. 3. Compare △s XSY and $X'S'Y'$. Auth.
Conclusion.

THEOREM V

112. (Converse of Theo. IV.) *If, in the same circle, or in equal circles, two chords are equal, the arcs subtended by them are equal.*

Hyp. In the equal ⓢ S and S', let chord $AB =$ chord $A'B'$.

To prove that $\frown AB = \frown A'B'$.

SUG. 1. Compare △s ASB and $A'S'B'$. Auth.
SUG. 2. Compare ∠s S and S'. Auth.
SUG. 3. Compare $\frown$s AB and $A'B'$. Auth.
Conclusion.

THEOREM VI

113. *In the same circle, or in equal circles, the greater of two unequal arcs, neither of which exceeds a semicircumference, is subtended by the greater chord.*

Note. Use arrangement for figures employed in Theos. II and III.

Hyp. In the equal $\circledS$ S and Y, let $\frown RT$ be $>\frown XZ$.

To prove that chord $RT >$ chord XZ.

Sug. 1. Draw radii RS, TS, XY, and ZY.

Sug. 2. Compare $\triangle$ RST and XYZ, referring to Bk. I, Theo. XXXVI.

Conclusion.

THEOREM VII

114. (Converse of Theo. VI.) *In the same circle, or in equal circles, the greater of two unequal chords subtends the greater arc, if neither arc exceeds a semicircumference.*

Hyp. In the equal $\circledS$ S and Y, let chord AB be $>$ chord CD.

To prove that $\frown AB > \frown CD$.

Sug. 1. Draw radii SA, SB, YC, and YD.

Sug. 2. Consult Bk. I, Theo. XXXVII.

Conclusion.

THEOREM VIII

115. *A diameter perpendicular to a chord bisects the chord and its subtended arcs.*

Hyp. In the $\odot C$, let the diameter DS be $\perp$ chord LM, intersecting LM at B.

To prove that $LB = BM$, also $\frown LD = \frown DM$ and $\frown LS = \frown SM$.

SUG. 1. Draw radii LC and MC.

SUG. 2. Compare $\triangle$s LBC and MBC. Auth.

First conclusion.

SUG. 3. $\angle LCB = \angle MCB$. Why? Therefore $\frown LD = \frown DM$. Auth. Also $\frown LS = \frown SM$. Auth.

Second conclusion.

EXERCISES

68. In the same circle, or in equal circles, equal central angles include equal sectors, and conversely equal sectors are included by equal central angles.

69. What is the locus of all points equidistant from the extremities of a chord?

THEOREM IX

116. *A line perpendicular to a chord at its middle point passes through the centre of the circle.*

Hyp. In the $\odot C$, let DX be $\perp$ chord AB at its middle pt. D.

To prove that DX passes through C.

SUG. 1. From C draw a line assumed to be $\perp AB$. This line must pass through D. Auth.

SUG. 2. Show that DX and DC must coincide. Auth.

Conclusion.

EXERCISES

70. Prove that the method used in Problems I and II, Bk. I, is correct.

71. Show how to bisect a given arc.

72. Show how to divide a given arc or angle into four equal parts.

73. A diameter passing through the middle point of a chord is perpendicular to the chord and bisects the subtended arcs.

74. A diameter drawn to the middle point of a given arc bisects the chord of the arc and is perpendicular to it.

75. If two chords of a circle cut each other and make equal angles with the straight line which joins their point of intersection to the centre, the chords are equal.

THEOREM X

117. *In the same circle, or in equal circles, equal chords are equidistant from the centre; and of two unequal chords the greater is nearer the centre.*

PART 1. **Hyp.** In the equal ⓢ C and S, let chord $BD =$ chord XZ. Also let CM be $\perp BD$ from C, $SW \perp XZ$ from S.

To prove that $CM = SW$.

SUG. 1. Draw radii CD and SZ.

SUG. 2. Compare MD and WZ. Auth.

SUG. 3. Compare △ CMD and SWZ. Auth.

Conclusion.

PART 2. **Hyp.** In the equal ⓢ C and S, let chord BH be $>$ chord XZ, and let CF be $\perp$ BH from C, SW drawn as in Hyp. of Part 1.

To prove that $SW > CF$.

SUG. 1. On ⌒ BH lay off ⌒ BD equal to ⌒ XZ. Draw $CM \perp$ chord BD from C, cutting chord BH at K.

SUG. 2. Compare CM and CK. Auth.

SUG. 3. Compare CK and CF. Auth.

SUG. 4. Therefore what relation has CM, or its equal SW, to CF? Auth.

Conclusion.

THEOREM XI

118. *A straight line perpendicular to a radius at its extremity is tangent to the circle.*

Hyp. Let CB be a radius of the $\odot C$ and XY be $\perp CB$ at its extremity B.

To prove that XY is tangent to $\odot C$ at B.

SUG. 1. From any other pt. in XY except B, as M, draw MC.

SUG. 2. $MC > BC$. Auth.

SUG. 3. Why does the statement in Sug. 2 prove XY tangent to $\odot C$?

Conclusion.

119. Problem XI. To draw a tangent to a given circle: first, from a point in the circumference; second, from a point without the circle.

THEOREM XII

120. *If a straight line is tangent to a circle, the radius meeting it at the point of contact is perpendicular to it.*

Hyp. Let XY be tangent to ⊙ C at Z, and let CZ be a radius.

To prove that $CZ \perp XY$.

SUG. See Bk. I, Theo. XXX, Cor.

Conclusion.

THEOREM XIII

121. *Two tangents to a circle from an outside point are equal.*

Hyp. Let PX and PY be tangents to the ⊙ C from an outside pt. P.

To prove that $PX = PY$.

SUG. 1. Draw PC, XC, and YC.

SUG. 2. Compare △ PXC and PYC.

Conclusion.

THEOREM XIV

122. *Arcs of a circle intercepted by parallel lines are equal.*

Hyp. In the $\odot C$, let chord PQ be $\parallel$ chord BD.

To prove that $\frown PB = \frown QD$.

SUG. 1. Draw the diameter $MS \perp PQ$, intersecting PQ at N, BD at R.

SUG. 2. What relation has MS to BD? Auth.

SUG. 3. $\frown PM = \frown QM$. Auth.

SUG. 4. $\frown BS = \frown DS$. Auth.

SUG. 5. $(\frown PM + \frown BS)$ has what relation to $\frown PB$? Also, $(\frown QM + \frown DS)$ has what relation to $\frown QD$?

SUG. 6. What follows regarding $\overset{\frown}{s}$ PB and QD? Why?

Conclusion.

SUG. 7. Prove also if one of the $\parallel$ lines is a tangent; if both are tangents.

NOTE. The method above applies whether the chords are on the same side or on opposite sides of the centre. A simpler method of proof also applies if the chords are both on the same side of the centre.

123. Cor. *The straight line joining the points of tangency of two parallel tangents is a diameter of the circle.*

THEOREM XV

124. *Through three points not in the same straight line, one circumference, and only one, can be drawn.*

Hyp. Let A, B, and C be three pts. not in same st. line.

To prove that one circumference, and only one, can be drawn through A, B, and C.

SUG. 1. Draw lines AB and BC, and erect a $\perp$ to each at its middle pt. Let these $\perp$s intersect at O. Draw OA, OB, and OC.

SUG. 2. Compare OA with OB; OB with OC. Auth.

SUG. 3. What relation, therefore, has O with A, B, and C?

SUG. 4. If one circumference can be drawn through A, B, and C, why cannot a second be drawn through the same pts.?

Conclusion.

125. Cor. *Two circles cannot intersect in more than two points.*

EXERCISES

76. What is the locus of the middle points of a series of parallel chords?

77. What is the locus of the middle points of a series of equal chords of a circle?

126. The **ratio** of one quantity to another of the same kind is the quotient of the first divided by the second.

The ratio of a to b is expressed as $\frac{a}{b}$ or $a:b$.

127. A **proportion** is the equality of two ratios.

128. Two quantities are **commensurable** when they have a common measure; that is, when a third quantity can be found which will be contained an integral number of times in each.

129. Two quantities are **incommensurable** when they have no common measure. The ratio of such quantities is called an **incommensurable ratio.**

An illustration of such a ratio is the ratio $\frac{\sqrt{2}}{4}$.

130. A **constant** is a quantity whose value remains fixed throughout a given discussion.

131. A **variable** is a quantity which may assume an indefinite number of values throughout a given discussion.

132. The **limit** of a variable is the constant from which the variable can be made to differ by less than any assigned value but to which it can never be made equal.

THEORY OF LIMITS

133. *If two variables are always equal as they approach their limits, the limits are equal.*

Two variables which are always equal may be considered as but one variable. They must necessarily approach the same limit.

NOTE. A more exhaustive explanation of the Theory of Limits may be given here at the discretion of the teacher.

THEOREM XVI

134. *In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.*

CASE I. *When the intercepted arcs are commensurable.*

Hyp. In the equal ⓢ C and C', let the central $\angle$s ACB and $DC'E$ intercept the commensurable arcs AB and DE respectively.

To prove that $\frac{\angle ACB}{\angle DC'E} = \frac{\frown AB}{\frown DE}$.

Proof. With a unit of measure common to $\frown$s AB and DE, divide $\frown AB$ into m equal parts, $\frown DE$ into n equal parts. Each division of $\frown AB =$ each division of $\frown DE$.

Then $\frac{\frown AB}{\frown DE} = \frac{m}{n}$.

Connect pts. of division on $\frown$s AB and DE, with C and C' respectively.

$\angle ACB$ is divided into m equal $\angle$s. Auth.

$\angle DC'E$ is divided into n equal $\angle$s. Auth.

Each division of $\angle ACB =$ each division of $\angle DC'E$.

$$\therefore \frac{\angle ACB}{\angle DC'E} = \frac{m}{n}.$$

$$\therefore \frac{\angle ACB}{\angle DC'E} = \frac{\frown AB}{\frown DE}. \quad \text{Auth.}$$

Case II. *When the intercepted arcs are incommensurable.*

Hyp. In the equal Ⓢ C and C', let the central $\angle$s ACB and $DC'F$ intercept the incommensurable $\frown$s AB and DF respectively.

To prove that $\dfrac{\angle C}{\angle C'}=\dfrac{\frown AB}{\frown DF}$.

Proof. Let $\frown AB$ be divided into any number of equal parts, and apply one of these parts to $\frown DF$. As $\frown$s AB and DF are incommensurable, $\frown DF$ will be divided into a certain number of equal parts with a remainder XF less than the divisor.

Draw XC'.

Then $\dfrac{\angle ACB}{\angle DC'X}=\dfrac{\frown AB}{\frown DX}$. Auth.

Consider the unit of measure of the $\frown AB$ to be indefinitely diminished. Then as the remainder must always be less than the divisor, XF will also indefinitely diminish.

Variable $\angle DC'X$ will gradually increase, approaching as its limit $\angle DC'F$.

Variable $\frown DX$ will approach as its limit $\frown DF$.

$\therefore$ variable ratio $\frac{\angle ACB}{\angle DC'X}$ will approach as its limit $\frac{\angle ACB}{\angle DC'F}$.

Also variable ratio $\frac{\frown AB}{\frown DX}$ will approach as its limit $\frac{\frown AB}{\frown DF}$.

$$\therefore \frac{\angle ACB}{\angle DC'F} = \frac{\frown AB}{\frown DF}. \quad \text{Auth.}$$

Q.E.D.

135. Scho. From the preceding theorem it may be said that a central angle *is measured by* its intercepted arc.

An **angle** of 40° intercepts, therefore, an **arc** of 40°.

A right angle intercepts an arc of 90°, commonly called a **quadrant.**

136. Problem XII. To find the centre of curvature of a given arc or circle.

Sug. See Bk. II, Theo. XV.

EXERCISES

78. How many degrees are there in a central angle which intercepts one-third of a circumference?

79. State the complement and supplement of an arc of 37°; of 53° 35′.

THEOREM XVII

137. *An inscribed angle is measured by one-half its intercepted arc.*

CASE I. *When one side of the angle is a diameter of the circle.*

Hyp. In the ⊙ *O*, let the inscribed ∠ *CAB* be formed by the chord *AC* and the diameter *AB*.

To prove that ∠ *CAB* is measured by $\frac{1}{2} \frown BC$.

SUG. 1. Draw radius *CO* and compare ∠s *CAB* and *ACO*. Auth.

SUG. 2. Compare the sum of the ∠s *CAB* and *ACO* with ∠ *COB*. Auth.

SUG. 3. Compare in magnitude ∠s *CAB* and *COB*.

SUG. 4. ∠ *COB* is measured by what arc? Auth. Therefore ∠ *CAB* is measured by what? Auth.

Conclusion.

CASE II. *When neither side of the inscribed angle is a diameter.*

SUG. Draw a diameter from the vertex of the angle, and apply Case I.

Conclusion.

138. Cor. *An angle inscribed in a semicircumference is a right angle.*

139. Problem XIII. To draw a circle tangent to a given circle.

THEOREM XVIII

140. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*

Hyp. Let AB be tangent to $\odot C$ at pt. D, and let DE be a chord of the circle.

To prove that $\angle BDE$ is measured by $\frac{1}{2} \frown DE$.

SUG. 1. Draw diameter DF.

SUG. 2. $\angle BDF$ is what kind of $\angle$, and is therefore measured by how much of semicircumference DF?

SUG. 3. $\angle EDF$ is measured by what? Auth.

SUG. 4. Therefore $\angle BDE$ is measured by what?

Conclusion.

THEOREM XIX

141. ***An angle formed by two intersecting chords is measured by one-half the sum of the intercepted arc and the arc intercepted by its vertical angle.***

Hyp. In $\odot ADBC$, chords AB and CD intersect at F.

To prove that $\angle AFC$ is measured by $\frac{1}{2}(\frown AC + \frown BD)$.

SUG. 1. Draw chord $CE \parallel AB$.

SUG. 2. Compare $\frown$s AC and BE. Auth.

SUG. 3. Compare $\angle$s AFC and FCE. Auth.

SUG. 4. But $\angle FCE$ is measured by one-half the sum of what arcs? Auth.

Conclusion.

EXERCISES

80. All angles inscribed in the same segments are equal.

81. Connect in succession any four points in the circumference of a circle and prove that either pair of opposite angles are supplementary.

82. An angle inscribed in a segment greater than a semicircle is acute; and an angle inscribed in a segment less than a semicircle is obtuse. Two chords perpendicular to a third chord at its extremities are equal.

THEOREM XX

142. *An angle formed by two secants intersecting without a circumference, by two tangents, or by a tangent and a secant, is measured by one-half the difference of the intercepted arcs.*

PART I. **Hyp.** From *A*, a pt. without the ⊙ *C*, let two secants be drawn intersecting the circumference in pts. *D* and *B*, *E* and *F*, respectively.

To prove that $\angle BAF$ is measured by $\frac{1}{2}(\frown BF - \frown DE)$.

SUG. 1. Draw chord $EM \parallel BA$.

SUG. 2. Method of proof similar to that of Theo. XIX.

PARTS II AND III. Proofs to be worked out by the pupil.

Conclusion.

NOTE. The space below is reserved for the figures for Theo. XX, Parts II and III.

EXERCISES

83. To construct an isosceles triangle, given the base and altitude.

SUG. See Bk. I, Theo. XXXI.

84. To construct an isosceles triangle, given the altitude and angle at the vertex.

85. To construct a parallelogram, given two diagonals and the angle to be included by them.

SUG. See Bk. I, Theo. XLIII.

86. To construct a right triangle, given the hypotenuse and an acute angle.

SUG. See Bk. I, Theo. XXIV, Cor. 2.

87. If two equal chords of a circle intersect, their corresponding segments are equal.

88. The lines joining the adjacent extremities of two equal chords of a circle are equal.

89. If through any point in a radius two chords be drawn making equal angles with it, these chords are equal.

90. The sum of the opposite sides of a quadrilateral described about a circle are equal.

91. Find a point that is equidistant from three points not in the same straight line.

92. Find the locus of the centre of a circle tangent to a given line at a given point.

93. Find the locus of the middle points of equal chords of a given circle.

94. Find the locus of the centre of a circle tangent to two parallel lines.

BOOK III

PROPORTIONAL LINES AND SIMILAR FIGURES

THEORY OF PROPORTION

Topics for definition.

143. Proportion.

Terms (*a*) Antecedents, Consequents.
(*b*) Extremes, Means.
Mean Proportional.

NOTE. In the suggestions under the following propositions, the *fractional* form of the ratio is referred to.

144. Proposition I. *In a proportion the product of the extremes equals the product of the means.*

SUG. Multiply each ratio by product of consequents.

145. Prop. II. *If the product of two quantities equals the product of two other quantities, the factors of one product may be made the means, the factors of the other the extremes, of a proportion.*

146. Prop. III. *Both terms of a ratio may be multiplied or divided by the same quantity without changing the value of the ratio.*

147. Prop. IV. *If four quantities are in proportion, they are in proportion by inversion.*

If $a : b = c : d$, then $b : a = d : c$.

SUG. Divide 1 by each ratio or apply Props. I and II.

148. Prop. V. *If four quantities are in proportion, they are in proportion by alternation.*

If $a : b = c : d$, then $a : c = b : d$.

SUG. Multiply each ratio by $\frac{b}{c}$.

149. Prop. VI. *If four quantities are in proportion, they are in proportion by composition.*

If $a : b = c : d$, then $a + b : b = c + d : d$ or $a + b : a = c + d : c$.

SUG. 1. To obtain the first conclusion, add 1 to each ratio.

SUG. 2. To obtain the second conclusion, treat the given proportion by inversion and then proceed as in Sug. 1.

150. Prop. VII. *If four quantities are in proportion, they are in proportion by division.*

If $a : b = c : d$, then $a - b : b = c - d : d$ or $a - b : a = c - d : c$.

SUG. 1. To obtain the first conclusion subtract 1 from each ratio.

SUG. 2. See Sug. 2, Prop. VI.

151. Prop. VIII. *If four quantities are in proportion, they are in proportion by composition and division.*

If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

SUG. Divide result of Prop. VI by that of Prop. VII.

152. Prop. IX. *In a series of equal ratios, the ratio of the sum of the antecedents to the sum of the consequents equals any one of the ratios.*

If $a : b = c : d = e : f$, then $a + c + e : b + d + f = a : b$.

SUG. 1. Let $r =$ each ratio. Then $\frac{a}{b} = r$, $\frac{c}{d} = r$, etc.

SUG. 2. Then $a = br$, $c = dr$, etc.

SUG. 3. By adding these equations $(a + c + e) = (b + d + f)r$.

SUG. 4. Divide by $(b + d + f)$.

153. Prop. X. *If three terms of a proportion are equal respectively to the corresponding terms of another proportion, the fourth terms are equal.*

THEOREM I

154. ***If parallel lines intercept equal segments on one transversal, they intercept equal segments on all transversals.***

Hyp. Let AE, BF, CG, and DH be $\parallel$ lines cut by transversal XY at A, B, C, and D respectively, and by transversal $X'Y'$ at E, F, G, and H respectively, and let $AB = BC = CD$.

To prove that $EF = FG = GH$.

SUG. 1. From E, draw $ER \parallel AB$ meeting BF at R. In like manner draw $FS \parallel BC$, $GT \parallel CD$.

SUG. 2. Compare ER with AB, FS with BC, etc., and from these results compare ER, FS, and GT.

SUG. 3. Compare $\triangle ERF$, FSG, and GTH.

Conclusion.

EXERCISES

95. Find a fourth proportional to 7, 25, and 35.

96. Find the third term of a proportion whose first, second, and fourth terms are 90, 170, and 510 respectively.

97. Find a mean proportional between 28 and 63.

98. Find a third proportional to 9 and 15.

THEOREM II

155. *A parallel to one side of a triangle divides the other two sides proportionally.*

CASE I. *When the segments of the two sides are commensurable.*

NOTE. Put figures for both parts of Theo. II in this space.

Hyp. In △ ABC, let DE be ‖ BC, intersecting AB in D, AC in E. Let segments AD and DB be commensurable.

To prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

SUG. 1. With a unit of measure common to AD and DB, divide AD into m equal parts, DB into n equal parts. Then $\frac{AD}{DB} = \frac{m}{n}$.

SUG. 2. Through these pts. of division draw lines ‖ BC. Then $\frac{AE}{EC} = \frac{m}{n}$. Auth.

Conclusion.

CASE II. *When the segments of the two sides are incommensurable.*

Hyp. In the △ RST, let BC be ‖ ST, and let segments RB and BS be incommensurable.

To prove that $\frac{RB}{BS} = \frac{RC}{CT}$.

Sug. 1. Divide RB into any number of equal parts, and apply one of these parts to BS. It will be contained in BS a certain number of times with a remainder XS less than the divisor. Draw $XY \parallel ST$.

Sug. 2. For remainder of proof, follow method outlined in Bk. II, Theo. XVI, Case II.

Conclusion.

156. Cor. *If a line is parallel to one side of a triangle, the ratio between either of the remaining sides and one of its segments equals the ratio between the other remaining side and its corresponding segment.*

THEOREM III

157. (Converse of Theo. II.) *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

Hyp. In $\triangle XYZ$, let DE divide XY and XZ in such a way that $\frac{XD}{DY}=\frac{XE}{EZ}$.

To prove that $DE \parallel YZ$.

Sug. 1. $\frac{XY}{XD}=\frac{XZ}{XE}$. Auth.

Sug. 2. Through D, draw DB, assuming it to be $\parallel YZ$, and apply Cor. under Theo. II.

Sug. 3. Compare proportion thus obtained with the proportion in Sug. 1. What follows regarding XE and XB? Auth. Regarding DE and DB? Auth.

Sug. 4. Why is DE therefore $\parallel YZ$?

Conclusion.

Topics for Definition.

158. Similar Polygons.

Homologous angles.
Homologous sides.

159. Scho. In similar polygons, the ratio between any two homologous lines is called the **ratio of similitude** of the polygons.

THEOREM IV

160. *If two triangles have their homologous angles equal, the triangles are similar.*

Hyp. In $\triangle$s ABC and $A'B'C'$, let $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

SUG. 1. Apply $\triangle ABC$ to $\triangle A'B'C'$ so that $\angle A$ coincides with $\angle A'$, and BC takes the position DE.

SUG. 2. What relation follows between DE and $B'C'$? Auth.

SUG. 3. $\dfrac{A'D}{A'B'} = \dfrac{A'E}{A'C'}$. Auth. $\therefore \dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.

SUG. 4. Apply $\triangle ABC$ again to $\triangle A'B'C'$ so that $\angle B$ coincides with $\angle B'$.

SUG. 5. By similar method of proof show that $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

Sug. 6. Using conclusions of Sug. 3 and Sug. 5, prove that the corresponding sides of the triangles are proportional.

Conclusion.

161. Cor. 1. *If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.*

162. Cor. 2. *If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.*

163. Cor. 3. *If a line is parallel to one side of a triangle, a smaller triangle is formed similar to the given triangle.*

Theorem V

164. ***If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.***

Hyp. In the $\triangle$ RST and $R'S'T'$, let $\angle R = \angle R'$ and let $\frac{RS}{R'S'} = \frac{RT}{R'T'}$.

To prove that $\triangle RST \sim$ to $\triangle R'S'T'$.

Sug. 1. Apply $\triangle RST$ to $\triangle R'S'T'$ so that $\angle R$ coincides with $\angle R'$ and ST assumes the position XY.

Sug. 2. Prove $XY \parallel S'T'$ and apply Theo. IV, Cor. 3.

Conclusion.

THEOREM VI

165. *If two triangles have their homologous sides proportional, the triangles are similar.*

Hyp. In $\triangle$s ABC and XYZ, let $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

To prove that $\triangle ABC \sim \triangle XYZ$.

SUG. 1. On XY take YA' equal to AB. On YZ take YC' equal to BC. Draw $A'C'$.

SUG. 2. $\triangle XYZ \sim \triangle A'YC'$. Auth.

SUG. 3. Obtain a proportion from the sides of these similar triangles, and compare it with the proportion in the Hyp. What follows regarding $A'C'$ and AC? Auth.

SUG. 4. Compare $\triangle$s $A'YC'$ and ABC. Auth.

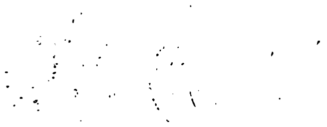
SUG. 5. Compare $\triangle$s ABC and XYZ.

Conclusion.

166. Problem XIV. To divide a given line into any number of equal parts.

THEOREM VII

167. *The ratio of homologous altitudes of similar triangles equals the ratio of similitude of the triangles.*



Hyp. Let $\triangle$ *ABC* and *RST* be similar, and let *AD* and *RM* be homologous altitudes.

To prove that $\frac{AD}{RM}=\frac{AB}{RS}$.

SUG. Compare $\triangle$ *ABD* and *RSM*. Auth.
Conclusion.

EXERCISES

99. Two isosceles triangles are similar if their vertical angles are equal.

100. Two isosceles triangles are similar if an angle of one is equal to an homologous angle of the other.

101. Two triangles are similar if their sides are respectively parallel.

102. Two triangles are similar if their sides are respectively perpendicular.

103. If the altitudes *AD* and *BW* of the triangle *ABC* intersect at *E*, triangle *AEW* is similar to triangle *BDE*.

104. The base and altitude of a triangle are 6 ft. and 3 ft. 5 in., respectively. The base of a similar triangle is 2 ft. 6 in.; find its altitude.

THEOREM VIII

168. ***The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.***

Hyp. Let AD bisect $\angle A$ of $\triangle ABC$.

To prove that $\frac{AC}{AB}=\frac{CD}{DB}$.

SUG. 1. Through B, draw a line $\parallel AD$ and meeting CA produced at K.

SUG. 2. Prove that $\triangle KAB$ is an isos. $\triangle$. What lines are therefore equal?

SUG. 3. Apply Theo. II. to the $\triangle KBC$.

SUG. 4. Modify this proportion by referring to the last conclusion obtained from Sug. 2.

Conclusion.

EXERCISE

105. The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides: thus, let the bisector of the exterior angle EAB of the triangle ACB meet the opposite side CB produced at D; to prove that $DB : DC = AB : AC$.

THEOREM IX

169. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*

Hyp. In polygons *ABCDE* and *RSTVW*, let △ *ABC*, *ACD*, and *ADE* be similar respectively to △ *RST*, *RTV*, and *RVW*, and similarly placed.

To prove that polygon *A–E* ∼ polygon *R–W*.

SUG. 1. By comparison of the similar △ in order, note that

$$\frac{AB}{RS}=\frac{BC}{ST}=\left(\frac{AC}{RT}\right)=\frac{CD}{TV}=\left(\frac{AD}{RV}\right)=\frac{DE}{VW}=\frac{AE}{RW}.$$

SUG. 2. By examining the hom. ∠ of the similar △, prove that the hom. ∠ of the polygons are equal.

Conclusion.

170. Problem XV. To construct a polygon similar to a given polygon on a given line homologous to one side of the given polygon.

THEOREM X

171. (Converse of Theo. IX.) *Two similar polygons are composed of the same number of triangles similar each to each and similarly placed.*

Hyp. Let A–E and R–W be two similar polygons.

To prove that $\triangle ABC$, ACD, and ADE are similar respectively to $\triangle RST$, RTV, and RVW, and similarly placed.

Sug. 1. In comparing exterior triangles of the polygons, as, for example, $\triangle ABC$ and RST, consult Theo. V.

Sug. 2. In comparing interior triangles, as $\triangle ACD$ and RTV, note that $\angle BCD = \angle STV$ (why ?), and that $\angle BCA = \angle STR$ (why ?), and from these equations show that $\angle ACD = \angle RTV$. In similar way, prove that $\angle ADC = \angle RVT$. Apply Theo. IV, Cor. 1.

Sug. 3. For polygons of more than 5 sides the method indicated in Sug. 2 for interior triangles will not apply. The proof of this case is left to the pupil.

Conclusion.

THEOREM XI

172. *The ratio of the perimeters of two similar polygons equals the ratio of similitude of the polygons.*

Hyp. Let $A\text{–}E$ and $R\text{–}W$ be two similar polygons and let P and P' represent their respective perimeters.

To prove that $P : P' = AB : RS$.

SUG. Apply Prop. IX to the continued proportion between the hom. sides of the polygons.

Conclusion.

173. Problem XVI. To divide a given line into parts proportional to any given lines.

THEOREM XII

174. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. *The triangles thus formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either side is a mean proportional between the whole hypotenuse and the adjacent segment.*

Hyp. In right $\triangle ABC$, let AD be $\perp BC$ from A, the vertex of the rt. $\angle$.

I. **To prove** that $\triangle ABD \sim \triangle ACD$, $\triangle ABD \sim \triangle ABC$, and $\triangle ACD \sim \triangle ABC$.

SUG. Consult Theo. IV, Cor. 2.

II. **To prove** that $\frac{BD}{AD}=\frac{AD}{DC}$.

SUG. Apply def. of similar polygons to $\triangle$s ABD and ACD.

III. **To prove** that $\frac{BC}{AB}=\frac{AB}{BD}$, also $\frac{BC}{AC}=\frac{AC}{CD}$.

SUG. Use method suggested under Part II of this theorem.

Conclusion.

THEOREM XIII

175. *If from a point without a circle two secants are drawn, the product of one secant and its external segment equals the product of the other and its external segment.*

Hyp. Let the two secants AC and AY intersect the circumference of a given ⊙ in points B and X, and C and Y, respectively.

To prove that $AC \times AB = AY \times AX$.

Sug. 1. Draw CX and BY.

Sug. 2. Compare △ AXC and ABY. Auth.

Sug. 3. Apply Prop. I to proportion obtained from previous suggestion.

Conclusion.

176. Cor. *If a tangent and a secant meet without a circle, the tangent is a mean proportional between the whole secant and its external segment.*

Sug. 1. Construct a circle O, and from an outside point A draw tangent AC, and secant AY intersecting circumference at X and Y. Draw CX and CY.

Sug. 2. Compare △ ACX and ACY.

THEOREM XIV

177. *If two chords of a circle intersect, the product of the segments of one equals the product of the segments of the other.*

Hyp. In ⊙ *ADBC*, let chords *AB* and *CD* intersect at *X*.

To prove that $AX \times XB = CX \times XD$.

SUG. 1. Draw *AD* and *BC*.
SUG. 2. Compare △ *AXD* and *BXC*. Auth.
SUG. 3. See Sug. 3 under Theo. XIII.
Conclusion.

178. Problem XVII. To find a fourth proportional to three given lines.

179. Problem XVIII. To find a mean proportional between two given lines.

EXERCISES

106. A secant from a given point without a circle is 1 ft. 6 in. long, and its external segment is 8 in. long. Find the length of a tangent from the same point.

107. If one side of a polygon is 2 ft. 6 in. long, what is the length of the homologous side of a similar polygon, if their perimeters are respectively 15 ft. and 25 ft.?

108. If the segments of the hypotenuse of a right triangle made by the perpendicular from the vertex of the right angle are 6 in. and 4 ft., find the length of the perpendicular and the length of each of the sides including the right angle.

109. The diagonals of a trapezoid divide each other proportionally.

SUG. Let AC and BD, diagonals of trapezoid $ABCD$, intersect at X. Prove $\triangle ABX \sim \triangle DCX$.

110. Two altitudes of a triangle are inversely proportional to the corresponding bases.

111. Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, and C, and the other in A', B', and C', respectively. Prove that $\triangle ABC \sim \triangle A'B'C'$.

BOOK IV

AREAS AND RATIOS OF POLYGONS

Topics for Definition.

180. Area of Polygon.

181. Altitude of Parallelogram.

THEOREM I

182. *Parallelograms having equal bases and equal altitudes are equivalent.*

Hyp. Let ▱s *ABCD* and *KLMN* have equal bases, *CD* and *MN*, and equal altitudes, *AX* and *KY*, respectively.

To prove that ▱ *ABCD* ≡ ▱ *KLMN*.

SUG. 1. Apply ▱ *ABCD* to ▱ *KLMN* so that *CD* coincides with *MN*, ▱ *ABCD* assuming the position of *A'B'MN*.

SUG. 2. Compare △s *A'KN* and *B'LM*. Auth.

SUG. 3. By subtracting each △ in turn from the whole figure prove the theorem.

Conclusion.

THEOREM II

183. *If two rectangles have equal altitudes, the ratio of their areas equals the ratio of their bases.*

CASE I. *When the bases are commensurable.*

Hyp. Let rects. $MNRD$ and $LHKW$ have equal altitudes, MD and LW, and let bases, DR and WK, be commensurable.

To prove that $\dfrac{\text{area of rect. } MR}{\text{area of rect. } LK} = \dfrac{DR}{WK}$.

SUG. 1. With a unit of measure common to DR and WK, divide DR into m equal parts, WK into n equal parts. Through these pts. of division draw lines $\parallel MD$ in rect. MR, and $\parallel LW$ in rect. LK.

SUG. 2. For remainder of proof, follow method outlined in Bk. II, Theo. XVI.

CASE II. *When the bases are incommensurable.*

Hyp. In rects. $M'R'$ and $L'K'$, let altitude $M'D' =$ altitude $L'W'$ and let bases $D'R'$ and $W'K'$ be incommensurable.

To prove that $\dfrac{\text{area of rect. } M'R'}{\text{area of rect. } L'K'} = \dfrac{D'R'}{W'K'}$.

SUG. 1. Apply a unit of measure of $W'K'$ to the line $D'R'$. Represent the remainder by YR'. Draw $XY \parallel M'D'$. Apply Case I and Theory of Limits.

Conclusion.

184. Cor. *If two rectangles have equal bases, the ratio of their areas equals the ratio of their altitudes.*

Note. In referring to plane figures, the words "area of" are often omitted. The conclusion in Theo. II might be expressed:

$$\frac{\text{rect. } MR}{\text{rect. } LK}=\frac{DR}{WK}.$$

Theorem III

185. *The ratio of the areas of two rectangles equals the ratio of the products of their bases and their altitudes.*

Hyp. Let R and S be two rects., and let a and b, a' and b', be their respective altitudes and bases.

To prove that $\dfrac{R}{S}=\dfrac{a \times b}{a' \times b'}$.

Sug. 1. Construct a third rect. T, having base b' and altitude a.
Sug. 2. Compare rect. R with rect. T. Auth.
Sug. 3. Compare rect. T with rect. S. Auth.
Sug. 4. Combine the results of the last two suggestions.
Conclusion.

THEOREM IV

186. *The area of a rectangle equals the product of its base and its altitude.*

Hyp. In the rect. $ABCD$, let b and a represent the base and altitude respectively.

To prove that area of rect. $ABCD = b \times a$.

SUG. Compare the area of rect. $ABCD$ with the unit of area, using Theo. III.

Conclusion.

THEOREM V

187. *The area of a parallelogram equals the product of its base and its altitude.*

Hyp. In the ▱ $ABCD$, let b and a represent the base and altitude respectively.

To prove that area of ▱ $ABCD = b \times a$.

SUG. 1. Construct a rect. $RSTW$ with same base and same altitude as rect. $ABCD$.

SUG. 2. Compare ▱ $ABCD$ with rect. $RSTW$. Auth.

Conclusion.

Theorem VI

188. ***The area of a triangle equals one-half the product of its base and altitude.***

Hyp. In $\triangle ABC$, let BC be the base, AX the altitude.

To prove that area of $\triangle ABC = \frac{1}{2} BC \times AX$.

Sug. 1. Draw $AD \parallel BC$, $CD \parallel AB$, making ▱ $ABCD$.

Sug. 2. See Bk. I, Theo. XXXVII, Cor., and Bk. IV, Theo. V.

Conclusion.

189. Cor. *The area of a trapezoid equals the product of its altitude and one-half the sum of its parallel sides.*

Sug. Draw diagonal and apply Theo. VI.

EXERCISES

112. The area of a triangle is 36 sq. ft. If its base is 9 ft., what is its altitude?

113. Upon the base of a rectangle 28 in. by 21 in. a triangle equivalent to the rectangle is constructed. What is the altitude of the triangle?

114. The area of a rhombus is equal to one-half the product of its diagonals.

115. The square upon the base of an isosceles triangle whose vertical angle is a right angle is equivalent to four times the triangle.

Sug. Draw the diagonals of the square.

THEOREM VII

190. *The ratio of the areas of two similar triangles equals the ratio of the squares of their corresponding sides or altitudes.*

Hyp. Let $\triangle ABC \sim \triangle DEF$, and let AM and DN be corresponding altitudes.

To prove that $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AM}^2}{\overline{DN}^2}$.

SUG. 1. By applying Theo. VI,

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} AM \times BC}{\frac{1}{2} DN \times EF}.$$

SUG. 2. $$\frac{\triangle ABC}{\triangle DEF} = \frac{AM \times BC}{DN \times EF} = \frac{AM}{DN} \times \frac{BC}{EF}.$$

SUG. 3. $$\frac{AM}{DN} = \frac{BC}{EF}. \quad \text{Auth.}$$

SUG. 4. Use conclusion of Sug. 3 in last part of equation in Sug. 2.

Conclusion.

EXERCISE

116. In two similar triangles, two homologous sides are 15 ft. and 25 ft. The area of the smaller triangle is 450 sq. ft. Find the area of the larger triangle.

THEOREM VIII

191. ***The ratio of the areas of two similar polygons equals the ratio of the squares of any corresponding sides.***

Hyp. Let $ABCDE$ and $A'B'C'D'E'$ be two similar polygons.

To prove that $\dfrac{\text{polygon } A\text{–}E}{\text{polygon } A'\text{–}E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2}$.

SUG. 1. Divide the polygons into a series of similar $\triangle$s by method used in Bk. III, Theo. X.

SUG. 2. $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{\triangle ACD}{\triangle A'C'D'} = \dfrac{\triangle ADE}{\triangle A'D'E'}$. Why?

SUG. 3. Apply Bk. III, Prop. IX, and Bk. IV, Theo. VII.
Conclusion.

EXERCISES

117. The area of a trapezoid is 52 sq. in., and the sum of the two parallel sides is 13 in. What is the distance between the parallel sides?

118. The area of a trapezoid is 30, the altitude is 5, and one base is 8. Find the other base.

119. $ABCD$ is a parallelogram and P is any point within it. Show that $\triangle PAB + \triangle PCD = \frac{1}{2} \square ABCD$.

THEOREM IX

192. *If two triangles have an angle of one equal to an angle of the other, the ratio of the areas of the triangles equals the ratio of the products of the sides including the equal angles.*

Hyp. In the $\triangle$ ABC and $A'B'C'$, let $\angle A = \angle A'$.

To prove that $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{AB \times AC}{A'B' \times A'C'}$.

SUG. 1. Apply $\triangle A'B'C'$ to $\triangle ABC$ so that $\angle A'$ coincides with $\angle A$. $\triangle A'B'C'$ assumes the position of $AB''C''$. Draw $B''C$.

SUG. 2. Compare $\triangle$ ABC and $AB''C$. What dimension have they in common?

SUG. 3. Compare $\triangle$ $AB''C$ and $AB''C''$. Auth.

SUG. 4. See Theo. III, Sug. 4.

Conclusion.

EXERCISES

120. The base and altitude of a triangle are 18 and 8 respectively. What is the altitude of a similar triangle whose base is 12?

121. A parallelogram is divided by its diagonals into four triangles of equal area.

122. If, in figure for Theo. IX, $AB = 3$, $AC = 4$, $A'B' = 5$, and $A'C' = 7$, what is the ratio of $\triangle ABC$ to $\triangle A'B'C'$?

Theorem X

193. ***The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.***

Hyp. Let ABC be a rt. $\triangle$, and let squares $ACDX$, $BCRF$, and $ABLK$ be described on the hypo. AC, side BC, and side AB respectively.

To prove that $AD \equiv BR + BK$.

Sug. 1. From B draw a line $BE \perp XD$. Draw BX and CK.

Sug. 2. In $\triangle$s KAC and BAX, $\angle KAC = \angle BAX$. Why? $\therefore \triangle KAC = \triangle BAX$. Auth.

Sug. 3. Compare areas of $\triangle KAC$ and square BK.

Sug. 4. Compare areas of $\triangle BAX$ and rect. AE.

Sug. 5. What follows regarding square BK and rect. AE?

Sug. 6. By similar method compare square BR with rect. CE.

Sug. 7. Add results of Sugs. 5 and 6.

Conclusion.

EXERCISES

123. Prove Theo. X algebraically by reference to Bk. III, Theo. XI, Part III.

124. Find area of right isosceles triangle whose hypotenuse is 100 yds. in length.

125. A man travels 25 miles east from a certain town, and another man 36 miles north. How far apart are they ?

194. Problem XIX. To construct a square equivalent to the sum of two given squares.

195. Problem XX. To construct a square equivalent to the difference of two given squares.

196. Problem XXI. To construct a square equivalent to a given parallelogram.

Sug. Find mean proportional between the two dimensions of the parallelogram.

EXERCISES

126. The altitude of an equilateral triangle is 60 in. How long are its sides?

127. Upon the diagonal of a rectangle 28 in. by 21 in. a triangle equivalent to the rectangle is constructed. What is the altitude of the triangle?

128. If lines are drawn perpendicular to the diagonals of a square at their extremities, a second square is formed equivalent to twice the original square.

129. Two parallel chords in a circle are each 8 ft. long, and the distance between them is 6 ft. Find the radius.

130. The hypotenuse of a right triangle is 5 ft. 5 in., and one of its legs is 2 ft. 9 in. Find its area.

131. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the homologous side of a similar polygon whose area is 392 sq. ft.?

132. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, prove triangle ABE equivalent to $\frac{1}{2}$ $ABCD$.

Sug. Through E draw a line $\parallel$ to AB.

133. The altitude of an equilateral triangle is 3. Find its area.

BOOK V

REGULAR POLYGONS AND THE CIRCLE

RATIOS AND MEASUREMENT

Topics for Definition.

197. **Regular polygon.**

198. **Polygon inscribed in a circle.**
Circle circumscribed about a polygon.

199. **Polygon circumscribed about a polygon.**
Circle inscribed in a polygon.

THEOREM I

200. *An equilateral polygon inscribed in a circle is a regular polygon.*

NOTE. **For convenience in constructing this figure, take the length of the radius as a side of the polygon.**

Hyp. Let the equilateral polygon *ABCDEF* be inscribed in the ⊙ *S*.

To prove that polygon *A–F* is equiangular and therefore regular.

SUG. Prove that the arcs intercepted by the ∡ *ABC, BCD,* etc., are equal. Then apply Bk. II, Theo. XVII.

Conclusion.

THEOREM II

201. ***A circle may be circumscribed about, or inscribed in, a regular polygon.***

Hyp. Let $ABCDEF$ be a regular polygon.

To prove that a circle may be circumscribed about, or inscribed in, the polygon A–F.

Sug. 1. Find a pt. O equidistant from A, B, and C. Auth. Draw AO, BO, CO, DO, etc.

Sug. 2. Prove $\triangle$s ABO and BCO equal and isos. Auth. $\therefore \angle ABO = \angle OBC = \angle BCO$.

Sug. 3. $\angle ABO = \frac{1}{2} \angle B$. Why? $\therefore \angle OCB = \frac{1}{2} \angle C$. Why?

Sug. 4. $\angle OCD = \frac{1}{2} \angle C = \angle OBC$. $\triangle BCO = \triangle CDO$. Auth. $\therefore OD = OC$.

Sug. 5. In same way prove E and F are the same distance from O as A, B, C, and D. First conclusion.

Sug. 6. Draw $OX \perp AB$, $OY \perp BC$, $OZ \perp CD$, etc., and apply Bk. II, Theo. X. Second conclusion.

Topics for Definition.

202. Parts of regular polygon.

Centre.	Apothem.
Radius.	Perimeter.

203. Cor. *The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

204. Problem XXII. To inscribe a regular hexagon in a circle.

Sug. See figure under Theo. I.

205. Problem XXIII. To inscribe a square in a circle.

EXERCISES

134. To inscribe a regular triangle or a regular dodecagon in a circle.

Sug. See Prob. XXII.

135. To inscribe a regular octagon in a circle.

Sug. See Prob. XXIII.

136. Find the angle at the centre of a regular pentagon.

137. Of a regular hexagon.

138. Of a regular octagon.

139. Of a regular decagon.

THEOREM III

206. *If the circumference of a circle be divided into any number of equal arcs, the chords subtending these arcs form a regular inscribed polygon; the tangents drawn at the points of division form a regular circumscribed polygon.*

PART I. **Hyp.** In the $\odot S$, let the equal $\frown$s AB, BC, CD, DE, and EA be subtended by the chords AB, BC, CD, etc.

To prove that the inscribed polygon A–E is regular.

SUG. Consult Theo. I.
First conclusion.

PART II. **Hyp.** At pts. A, B, C, D, and E, on the circumference of $\odot S$, let tangents be drawn intersecting at F, G, H, K, and M.

To prove that circumscribed polygon F–M is regular.

SUG. 1. Prove $\triangle$s AFB, BGC, etc., to be equal and isos. $\triangle$s, using Bk. II, Theo. XVIII.
SUG. 2. From Sug. 1, $\angle F = \angle G = \angle H$, etc., and $FB + BG = GC + CH$, etc.
Second conclusion.

207. Cor. 1. *If the vertices of a regular inscribed polygon are connected with the middle points of the arcs subtended by the sides of the polygon, a regular inscribed polygon of double the number of sides is formed.*

208. Cor. 2. *If a regular polygon be circumscribed about a circle, and if at the middle points of the arcs between the points of tangency tangents are drawn, a regular circumscribed polygon of double the number of sides is formed.*

EXERCISES

140. Find the value of an angle at a vertex of a regular hexagon; of a regular octagon; of a regular decagon; of a regular dodecagon.

141. A central angle of any regular polygon is the supplement of an angle of the polygon.

THEOREM IV

209. ***Regular polygons of the same number of sides are similar.***

Hyp. Let A and A' be two regular polygons each having n sides.

To prove that polygon A is similar to polygon A'.

SUG. 1. Since the polygons are equiangular, divide the sum of the int. ∡s of each polygon by the number of its ∡s. What does this prove?

SUG. 2. Write a proportion of the sides of the two polygons taken in order, and prove this proportion to be correct.

Conclusion.

EXERCISES

142. The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.

143. The perimeters of regular polygons of the same number of sides are to each other as any two homologous sides.

144. Find the area of a square inscribed in a circle whose radius is 6.

145. Show that the apothem of a regular inscribed hexagon $= \frac{r}{2}\sqrt{3}$.

146. Show that the side of an inscribed square $= r\sqrt{2}$.

147. The sides of a regular circumscribed polygon are bisected at the points of tangency.

THEOREM V

210. *The ratio of the perimeters of two regular polygons of the same number of sides equals the ratio of their radii or apothems.*

Hyp. In the regular n-sided polygons A–E and A'–E', let AO and $A'O'$, XO and $X'O'$ represent their respective radii and apothems, P and P' their respective perimeters.

To prove that $\frac{P}{P'} = \frac{AO}{A'O'} = \frac{XO}{X'O'}$.

SUG. Draw radii OB and $O'B'$, and compare △ AOB and $A'O'B'$. Consider OX and $O'X'$ as the respective altitudes of these △, and consult Bk. III, Theos. VII and XI.

Conclusion.

211. Cor. *The ratio of the areas of two regular polygons of the same number of sides equals the ratio of the squares of their radii or apothems.*

SUG. See Bk. IV, Theo. VIII.

EXERCISES

148. The apothem of a regular pentagon is 4.1 in. and a side is 6 in. Find the side of a regular pentagon whose apothem is 8.2.

149. One side of a polygon is 2 ft. 6 in. long. What is the length of the corresponding side of a similar polygon, the perimeters being, respectively, 20 ft. and 30 ft.?

THEOREM VI

212. *The area of a regular polygon equals half the product of its perimeter and its apothem.*

Hyp. Let A be a regular polygon on n sides, P its perimeter, and R its apothem.

To prove that area of $A = \frac{1}{2} P \times R$.

SUG. Draw the radii of the polygon, and apply Bk. IV, Theo. VI.

Conclusion.

EXERCISES

150. The area of any polygon that is circumscribed about a circle is equal to half the product of its perimeter and the radius of the circle.

151. Find the area of a regular hexagon whose side is 10 ft.

152. The apothem of a regular pentagon is 6 and a side is 4. Find the perimeter and area of a regular pentagon whose apothem is 8.

153. $ABCDE$ is a regular pentagon. If the diagonals AC and BE intersect at F, prove that $CDEF$ is a parallelogram.

154. The radius of a circle is 10; find the perimeter and area of the regular inscribed octagon.

155. Show that the side of an inscribed equilateral triangle $= r\sqrt{3}$.

THEOREM VII

213. *If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem of the polygon approaches the radius of the circle as its limit.*

Hyp. Let AB be a side and OC the apothem of a regular polygon of n sides inscribed in a circle whose radius is OA.

To prove that OC approaches OA as a limit as n increases indefinitely.

SUG. 1. $OA < OC + AC$. Auth.
SUG. 2. $OA - OC < AC$. Why?
$\therefore$ $OA - OC < 2\,AC$ or AB.
SUG. 3. As n increases indefinitely, AB decreases indefinitely.
$\therefore$ $OA - OC$ must approach 0 as its limit.
Conclusion.

EXERCISE

156. The area of an inscribed regular octagon is equal to the product of the lengths of the sides of the inscribed and circumscribed squares.

THEOREM VIII

214. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; and the area of the circle is the limit of the areas of these polygons.*

Hyp. Let P and P' be the perimeters and R and R' the apothems, S and S' the areas, respectively, of two circumscribed and inscribed polygons of the same number of sides.

I. **To prove** that the circumference of the circle is the common limit of P and P' when the number of sides is indefinitely increased.

SUG. 1. $\frac{P}{P'}=\frac{R}{R'}$. Auth. Then $\frac{P-P'}{P'}=\frac{R-R'}{R'}$. Auth.

$\therefore\ P-P'=\frac{P'(R-R')}{R'}$.

SUG. 2. By indefinitely doubling the number of sides show that $R-R'$ and $\frac{P'}{R'}(R-R')$ approaches 0 as its limit.

SUG. 3. $\therefore$ $P-P'$ must approach 0 at its limit; that is, P and P' must approach each other; hence each will approach indefinitely the circumference of the circle.

Conclusion.

II. **To prove** that the area of the circle is the common limit of S and S' when the number of sides is indefinitely increased.

Sug. 4. $\frac{S}{S'}=\frac{(R)^2}{(R')^2}$. Auth. See Part I for method.

Conclusion.

Theorem IX

215. *The circumferences of two circles have the same ratio as their radii.*

Hyp. Let C and C' be the circumferences, R and R' the radii, of two circles O and O'.

To prove that $\frac{C}{C'}=\frac{R}{R'}$.

Sug. 1. Inscribe in the ⓢ two regular polygons of the same number of sides. Then $\frac{P}{P'}=\frac{R}{R'}$. Auth.

Sug. 2. $P=P'\times\frac{R}{R'}$.

Sug. 3. By theory of limits show that $C=C'\times\frac{R}{R'}$, and therefore that $\frac{C}{C'}=\frac{R}{R'}$.

216. Cor. *The areas of two circles have the same ratio as the squares of their radii.*

Sug. See Theo. VIII, Part II.

217. Scho. The ratio of the circumference of a circle to its diameter is a fixed quantity. This ratio is designated by π. Hence $\frac{c}{d} = \pi$, and $\therefore C = \pi d = 2\pi r$. The numerical value of π is approximately 3.14159 +, or about $3\frac{1}{7}$.

Note. See Problem XXV, page 108, for proof regarding value of π.

Theorem X

218. ***The area of a circle equals half the product of the circumference and the radius.***

Sug. Inscribe a regular polygon, and prove that $\frac{1}{2} P \times a$ (apothem) approaches as its limit $\frac{1}{2} C \times r$.

219. Scho. If $C = 2\pi r$, then $\frac{1}{2} C \times r$ (the area of circle) $= \frac{1}{2}(2\pi r) \times r = \pi r^2$.

EXERCISES

157. Similar arcs are to each other as their radii, and similar sectors as the squares of their radii.

158. If the radius of a circle is 4, find its circumference and area.

159. If the circumference of a circle is 30, find its radius and area.

160. If the diameter of a circle is 26, find the length of an arc of 90°.

220. Problem XXIV. Given the radius and side of a regular inscribed polygon, to compute the side of a regular inscribed polygon of double the number of sides.

Hyp. In ⊙ *O*, let *AB* be the side of a regular inscribed polygon and let the radius *OX* be ⊥ to *AB*, intersecting chord *AB* at *Y* and ⌒ *AB* at *X*.

To find value of *AX*, the side of a regular inscribed polygon of double the number of sides.

Draw *AO*. In rt. △ *AOY*,

$$\overline{OY}^2 = \overline{AO}^2 - \overline{AY}^2. \quad \text{Auth.} \quad \text{But } AY = \tfrac{1}{2} AB. \quad \text{Auth.}$$

$$\therefore \; OY = \sqrt{r^2 - \frac{\overline{AB}^2}{4}} = \tfrac{1}{2}\sqrt{4r^2 - \overline{AB}^2}.$$

$$XY = OX - OY = r - \tfrac{1}{2}\sqrt{4r^2 - \overline{AB}^2}.$$

$$\therefore \text{ In rt. } \triangle AXY,\ \overline{AX}^2 = \overline{AY}^2 + \overline{XY}^2 = \frac{\overline{AB}^2}{4} + \left(r - \tfrac{1}{2}\sqrt{4r^2 - \overline{AB}^2}\right)^2.$$

$$\overline{AX}^2 = 2r^2 - r\sqrt{4r^2 - \overline{AB}^2}.$$

$$\therefore \; AX = \sqrt{r\left(2r - \sqrt{4r^2 - \overline{AB}^2}\right)}.$$

Q.E.F.

221. Cor. *If $r = 1$ and if the side of the inscribed polygon is represented by s, the side of the inscribed polygon of double the number of sides equals* $\sqrt{2 - \sqrt{4 - s^2}}$.

EXERCISES

161. Find the radius of a circle whose area is 6 sq. in.

162. What is the area of a circle circumscribed about a square whose side is 3?

222. Problem XXV. To compute, approximately, the value of π.

Hyp. Let c and d represent, respectively, the circumference and diameter of a circle whose radius is unity.

To compute the value of π or $\frac{c}{d}$.

Since the perimeter of a regular inscribed polygon, by indefinitely increasing the number of its sides, approaches the circumference of the circle as its limit, it follows that

$$\frac{\text{perimeter of polygon}}{\text{diameter of circle}} \text{ approaches as its limit } \frac{c}{d},$$

as the number of sides of the polygon is indefinitely increased.

Let s_6, s_{12}, etc., represent the side of a regular hexagon, of a regular dodecagon, etc., p_6, p_{12}, etc., their perimeters,

Then $$s_{12} = \sqrt{2-\sqrt{4-1^2}} = .51763809; \frac{p_{12}}{d} = 3.10582854,$$

$$s_{24} = \sqrt{2-\sqrt{4-(.51763809)^2}} = .26105238; \frac{p_{24}}{d} = 3.13262861,$$

$$s_{48} = \sqrt{2-\sqrt{4-(.26105238)^2}} = .13080626; \frac{p_{48}}{d} = 3.13935020,$$

and by continuing the process

$$s_{768} = .00818121; \frac{p_{768}}{d} = 3.14158470.$$

$\therefore \pi = 3.14159$, approximately. Q.E.F.

EXERCISES

163. How many degrees are there in an arc 18 in. long of a circumference whose radius is 5 ft.?

164. The ratio of the radii of two similar segments is 3 : 5. What is the ratio of their areas?

165. The circumference of a circle is 8. Find the circumference of a circle having twice the area of the given circle.

166. A circle has an area of 60 sq. in. What is the length of an arc of 40°?

167. The side of a square is 8. Find the circumferences of its inscribed and circumscribed circles.

168. A circular grass plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

INDEX TO PROBLEMS OF CONSTRUCTION

PAGE
Problem I. To draw a perpendicular to a straight line at some point in it 6
Problem II. To bisect a given angle 6
Problem III. To construct an angle equal to a given angle 10
Problem IV. To bisect a given straight line . . . 12
Problem V. Given two sides and the angle to be included by them, to construct the triangle . . 16
Problem VI. Given two angles and the side to be included by them, to construct the triangle . . 18
Problem VII. Given three sides of a triangle, to construct the triangle 19
Problem VIII. To draw a line parallel to a given line through a given point 19
Problem IX. To draw a line perpendicular to a given line from an outside point 20
Problem X. Given two angles of a triangle, to find the third 31
Problem XI. To draw a tangent to a given circle . . 54
Problem XII. To find the centre of curvature of a given arc or circle 61
Problem XIII. To draw a circle tangent to a given circle . 63
Problem XIV. To divide a given line into any number of equal parts 74

PAGE

Problem XV. To construct a polygon similar to a given polygon on a given line homologous to one side of the given polygon . . . 77

Problem XVI. To divide a given line into parts proportional to any given lines 79

Problem XVII. To find a fourth proportional to three given lines 82

Problem XVIII. To find a mean proportional between two given lines 83

Problem XIX. To construct a square equivalent to the sum of two given squares 93

Problem XX. To construct a square equivalent to the difference of two given squares . . . 93

Problem XXI. To construct a square equivalent to a given parallelogram 94

Problem XXII. To inscribe a regular hexagon in a circle . 97

Problem XXIII. To inscribe a square in a circle . . . 97

SYLLABUS OF PLANE GEOMETRY

BOOK I

17. **Theorem I.** From a point in a straight line one perpendicular to the line, and only one, can be drawn.

19. **Theorem II.** All right angles are equal.

21. **Theorem III.** If one straight line meets another, the sum of the adjacent angles thus formed equals two right angles.

22. *Corollary* 1. The sum of all the angles about a point on one side of a straight line equals two right angles.

23. *Corollary* 2. The sum of all the angles about a point equals four right angles.

24. *Corollary* 3. Complements of the same angle or of equal angles are equal.

25. *Corollary* 4. Supplements of the same angle or of equal angles are equal.

26. **Theorem IV.** If two straight lines intersect, they form vertical angles which are equal.

27. **Theorem V.** If two adjacent angles are together equal to two right angles, their exterior sides are in the same straight line.

29. **Theorem VI.** If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal.

32. **Theorem VII.** If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal.

34. **Theorem VIII.** The angles opposite the equal sides of an isosceles triangle are equal.

35. *Corollary.* An equilateral triangle is also equiangular.

36. **Theorem IX.** If two triangles have three sides of one equal respectively to three sides of the other, the triangles are equal.

37. **Theorem X.** If a perpendicular be erected at the middle point of a straight line, the distances from any point in the perpendicular to the extremities of the line are equal.

39. **Theorem XI.** If a perpendicular be erected at the middle point of a straight line, the distances from any point not in the perpendicular to the extremities of the line are unequal.

41. **Theorem XII.** From a point without a straight line, one perpendicular to the line, and only one, can be drawn.

42. **Theorem XIII.** If two right triangles have the hypotenuse and an acute angle of one equal respectively to the hypotenuse and an acute angle of the other, the triangles are equal.

44. **Theorem XIV.** Two lines perpendicular to the same line are parallel.

48. **Theorem XV.** A line perpendicular to one of two parallel lines is perpendicular to the other also.

49. *Corollary.* Two lines parallel to a third line are parallel to each other.

52. **Theorem XVI.** If two parallel lines are cut by a transversal, the alternate-interior angles are equal.

53. **Theorem XVII.** If two parallel lines are cut by a transversal, the corresponding angles are equal.

54. *Corollary.* If two parallel lines are cut by a transversal, the alternate-exterior angles are equal.

55. **Theorem XVIII.** If two parallel lines are cut by a transversal, the sum of the interior angles on the same side of the transversal equals two right angles.

56. *Corollary.* If two parallel lines are cut by a transversal, the sum of the exterior angles on the same side of the transversal equals two right angles.

57. **Theorem XIX.** (Converse of Theo. XVI.) If two straight lines are cut by a transversal, so that the alternate-interior angles are equal, the lines are parallel.

58. **Theorem XX.** (Converse of Theo. XVII.) If two straight lines are cut by a transversal, so that the corresponding angles are equal, the lines are parallel.

59. *Corollary.* If two straight lines are cut by a transversal, so that the alternate-exterior angles are equal, the lines are parallel.

60. **Theorem XXI.** (Converse of Theo. XVIII.) If two straight lines are cut by a transversal, so that the sum of the interior angles on the same side of the transversal equals two right angles, the lines are parallel.

61. *Corollary.* If two straight lines are cut by a transversal, so that the sum of the exterior angles on the same side of the transversal equals two right angles, the lines are parallel.

62. **Theorem XXII.** Angles whose sides are parallel, each to each, are either equal or supplementary.

64. *Corollary.* Angles whose sides are perpendicular, each to each, are either equal or supplementary.

66. **Theorem XXIII.** An exterior angle of a triangle equals the sum of the opposite interior angles.

67. *Corollary.* An exterior angle of a triangle is greater than either of the opposite interior angles.

68. **Theorem XXIV.** The sum of the interior angles of a triangle equals two right angles.

69. *Corollary* 1. In a triangle there can be only one obtuse or one right angle.

70. *Corollary* 2. Every right triangle has two acute angles, each of which is the complement of the other.

71. *Corollary* 3. If two triangles have two angles of one equal to two angles of the other, the third angles are equal.

72. *Corollary* 4. If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.

73. **Theorem XXV.** If two right triangles have a side and an acute angle of one equal respectively to a side and an acute angle of the other, the triangles are equal.

74. **Theorem XXVI.** If two triangles have two angles of one equal respectively to two angles of the other and the sides opposite a pair of equal angles equal the triangles are equal.

75. **Theorem XXVII.** (Converse of Theo. VIII.) If two angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.

76. *Corollary.* An equiangular triangle is also equilateral.

77. **Theorem XXVIII.** If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.

78. *Corollary.* The hypotenuse is the greatest side of a right triangle.

80. **Theorem XXIX.** (Converse of Theo. XXVIII.) If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle being opposite the greater side.

81. **Theorem XXX.** A perpendicular is the shortest line that can be drawn from a point to a line.

82. *Corollary.* The shortest distance from a point to a line is the perpendicular from the point to the line.

83. **Theorem XXXI.** If a perpendicular and oblique lines be drawn from a point to a line:

I. Two oblique lines which meet the given line at equal distances from the foot of the perpendicular, are equal.

II. Of two oblique lines meeting the given line at unequal distances from the foot of the perpendicular, the more remote is the greater.

84. **Theorem XXXII.** Two equal oblique lines drawn from a point in a perpendicular to a given line, cut off equal distances from the foot of the perpendicular.

85. **Theorem XXXIII.** Two unequal oblique lines drawn from a point in a perpendicular to a given line, cut off unequal distances from the foot of the perpendicular, the longer line cutting off the greater distance.

86. **Theorem XXXIV.** If two right triangles have the hypotenuse and a side of one equal respectively to the hypotenuse and a side of the other, the triangles are equal.

87. **Theorem XXXV.** If two lines be drawn from a point in a triangle to the extremities of a side, the sum of these lines is less than the sum of the other two sides of the triangle.

88. **Theorem XXXVI.** If two triangles have two sides of one equal to two sides of the other and the included angles unequal, the remaining sides are unequal, that being greater which belongs to the triangle having the greater included angle.

89. **Theorem XXXVII.** If two triangles have two sides of one equal to two sides of the other and the third sides unequal, the angles opposite the third sides are unequal, that being greater which is opposite the greater third side.

92. **Theorem XXXVIII.** The opposite sides of a parallelogram are equal.

93. *Corollary.* The diagonal of a parallelogram divides it into two equal triangles.

94. **Theorem XXXIX.** The opposite angles of a parallelogram are equal.

95. **Theorem XL.** If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.

96. **Theorem XLI.** A quadrilateral whose opposite sides are equal is a parallelogram.

97. **Theorem XLII.** Two parallelograms having two sides and

the included angle of one equal to two sides and the included angle of the other are equal.

98. **Theorem XLIII.** The diagonals of a parallelogram bisect each other.

99. **Theorem XLIV.** The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, minus four right angles.

100. *Corollary.* In an equiangular polygon of n sides, the value of each angle equals $\frac{(2n-4) \text{ rt. } \measuredangle s}{n}$.

101. **Theorem XLV.** If the sides of a polygon be produced in succession, the sum of the exterior angles thus formed equals four right angles.

BOOK II

106. **Theorem I.** The diameter of a circle is greater than any other chord; and it bisects the circle and the circumference.

108. **Theorem II.** In the same circle, or in equal circles, equal central angles intercept equal arcs.

109. **Theorem III.** (Converse of Theo. II.) In the same circle, or in equal circles, equal arcs subtend equal central angles.

111. **Theorem IV.** If, in the same circle, or in equal circles, two arcs are equal, the chords subtending them are equal.

112. **Theorem V.** (Converse of Theo. IV.) If, in the same circle, or in equal circles, two chords are equal, the arcs subtended by them are equal.

113. **Theorem VI.** In the same circle, or in equal circles, the greater of two unequal arcs, neither of which exceeds a semicircumference, is subtended by the greater chord.

114. **Theorem VII.** (Converse of Theo. VI.) In the same circle, or in equal circles, the greater of two unequal chords subtends the greater arc, if neither arc exceeds a semicircumference.

115. **Theorem VIII.** A diameter perpendicular to a chord bisects the chord and its subtended arcs.

116. **Theorem IX.** A line perpendicular to a chord at its middle point passes through the centre of the circle.

117. **Theorem X.** In the same circle, or in equal circles, equal chords are equidistant from the centre; and of two unequal chords the greater is nearer the centre.

118. **Theorem XI.** A straight line perpendicular to a radius at its extremity is tangent to the circle.

120. **Theorem XII.** If a straight line is tangent to a circle, the radius meeting it at the point of contact is perpendicular to it.

121. **Theorem XIII.** Two tangents to a circle from an outside point are equal.

122. **Theorem XIV.** Arcs of a circle intercepted by parallel lines are equal.

123. *Corollary.* The straight line joining the points of tangency of two parallel tangents is a diameter of the circle.

124. **Theorem XV.** Through three points not in the same straight line one circumference, and only one, can be drawn.

125. *Corollary.* Two circles cannot intersect in more than two points.

134. **Theorem XVI.** In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.

137. **Theorem XVII.** An inscribed angle is measured by one-half its intercepted arc.

138. *Corollary.* An angle inscribed in a semicircumference is a right angle.

140. **Theorem XVIII.** An angle formed by a tangent and a chord is measured by one-half the intercepted arc.

141. **Theorem XIX.** An angle formed by two intersecting chords is measured by one-half the sum of the intercepted arc and the arc intercepted by its vertical angle.

142. **Theorem XX.** An angle formed by two secants intersecting without a circumference, by two tangents, or by a tangent and a secant, is measured by one-half the difference of the intercepted arcs.

BOOK III

154. **Theorem I.** If parallel lines intercept equal segments on one transversal, they intercept equal segments on all transversals.

155. **Theorem II.** A parallel to one side of a triangle divides the other two sides proportionally.

156. *Corollary.* If a line is parallel to one side of a triangle, the ratio between either of the remaining sides and one of its segments equals the ratio between the other remaining side and its corresponding segment.

157. **Theorem III.** (Converse of Theo. II.) If a line divides two sides of a triangle proportionally, it is parallel to the third side.

160. **Theorem IV.** If two triangles have their homologous angles equal, the triangles are similar.

161. *Corollary* 1. If two triangles have two angles of one equal to two angles of the other, the triangles are similar.

162. *Corollary* 2. If two right triangles have an acute angle of one equal to an acute angle of the other, the triangles are similar.

163. *Corollary* 3. If a line is parallel to one side of a triangle, a smaller triangle is formed similar to the given triangle.

164. **Theorem V.** If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.

165. **Theorem VI.** If two triangles have their homologous sides proportional, the triangles are similar.

167. **Theorem VII.** The ratio of homologous altitudes of similar triangles equals the ratio of similitude of the triangles.

168. **Theorem VIII.** The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

169. **Theorem IX.** If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.

171. **Theorem X.** (Converse of Theo. IX.) Two similar polygons are composed of the same number of triangles similar each to each and similarly placed.

172. **Theorem XI.** The ratio of the perimeters of two similar polygons equals the ratio of similitude of the polygons.

174. **Theorem XII.** If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:

I. The triangles thus formed are similar to the whole triangle, and to each other.

II. The perpendicular is a mean proportional between the segments of the hypotenuse.

III. Either side is a mean proportional between the whole hypotenuse and the adjacent segment.

175. **Theorem XIII.** If from a point without a circle two secants are drawn, the product of one secant and its external segment equals the product of the other and its external segment.

176. *Corollary.* If a tangent and a secant meet without a circle, the tangent is a mean proportional between the whole secant and its external segment.

177. **Theorem XIV.** If two chords of a circle intersect, the product of the segments of one equals the product of the segments of the other.

BOOK IV

182. **Theorem I.** Parallelograms having equal bases and equal altitudes are equivalent.

183. **Theorem II.** If two rectangles have equal altitudes, the ratio of their areas equals the ratio of their bases.

184. *Corollary.* If two rectangles have equal bases, the ratio of their areas equals the ratio of their altitudes.

185. **Theorem III.** The ratio of the areas of two rectangles equals the ratio of the products of their bases and their altitudes.

186. **Theorem IV.** The area of a rectangle equals the product of its base and its altitude.

187. **Theorem V.** The area of a parallelogram equals the product of its base and its altitude.

188. **Theorem VI.** The area of a triangle equals one-half the product of its base and altitude.

189. *Corollary.* The area of a trapezoid equals the product of its altitude and one-half the sum of its parallel sides.

190. **Theorem VII.** The ratio of the areas of two similar triangles equals the ratio of the squares of their corresponding sides or altitudes.

191. **Theorem VIII.** The ratio of the areas of two similar polygons equals the ratio of the squares of any corresponding sides.

192. **Theorem IX.** If two triangles have an angle of one equal to an angle of the other, the ratio of the areas of the triangles equals the ratio of the products of the sides including the equal angles.

193. **Theorem X.** The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.

BOOK V

200. **Theorem I.** An equilateral polygon inscribed in a circle is a regular polygon.

201. **Theorem II.** A circle may be circumscribed about, or inscribed in, a regular polygon.

203. *Corollary.* The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.

206. **Theorem III.** If the circumference of a circle be divided into any number of equal arcs, the chords subtending these arcs form a regular inscribed polygon; the tangents drawn at the points of division form a regular circumscribed polygon.

207. *Corollary* 1. If the vertices of a regular inscribed polygon are connected with the middle points of the arcs subtended by the sides of the polygon, a regular inscribed polygon of double the number of sides is formed.

208. *Corollary* 2. If a regular polygon be circumscribed about a circle, and if at the middle points of the arcs between the points of tangency tangents are drawn, a regular circumscribed polygon of double the number of sides is formed.

209. **Theorem IV.** Regular polygons of the same number of sides are similar.

210. **Theorem V.** The ratio of the perimeters of two regular polygons of the same number of sides equals the ratio of their radii or apothems.

211. *Corollary.* The ratio of the areas of two regular polygons of the same number of sides equals the ratio of the squares of their radii or apothems.

212. **Theorem VI.** The area of a regular polygon equals half the product of its perimeter and its apothem.

213. **Theorem VII.** If the number of sides of a regular inscribed polygon be increased indefinitely, the apothem of the polygon approaches the radius of the circle as its limit.

214. **Theorem VIII.** The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is increased indefinitely; and the area of the circle is the limit of the areas of these polygons.

215. **Theorem IX.** The circumferences of two circles have the same ratio as their radii.

216. *Corollary.* The areas of two circles have the same ratio as the squares of their radii.

218. **Theorem X.** The area of a circle equals half the product of the circumference and the radius.

This book should be returned to the Library on or before the last date stamped below.

A fine of five cents a day is incurred by retaining it beyond the specified time.

Please return promptly.

~~DUE JUN -8 '33~~